Chapter 2. Hamiltonian Mechanics

2.1. Lagrangian Formulation of Mechanics.

The lagrangian
\[ L = L(q_i, \dot{q}_i, t) \]

The integral action.
\[ S[q] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) \, dt. \]

is required to be a "minimum" (a critical point).

The increment of the action is given by:
\[ \Delta S[q] = S[q+h] - S[q] \]

\[ = \int_{t_1}^{t_2} \left[ L(q_i+h_i, \dot{q}_i+h_i, t) - L(q_i, \dot{q}_i, t) \right] \, dt. \]

\[ = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} h_i + \frac{\partial L}{\partial \dot{q}} \dot{h}_i \right] \, dt + o(h_i^2). \]

Integrating by parts:
\[ = SS \quad \text{is the 1st variation of} \]
\[ = \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \, dh_i + o(h_i^2). \]

It can be proved that
\[ SS = 0 \iff \forall h_i \in C^1([t_1, t_2]). \]

Euler-Lagrange equations.
Tabor assumes that Landau and Lifshitz claim that, "for a free particle "at least," the principles of homogeneity in time and isotropy of space determine that the Lagrangian can only be proportional to the square of the (generalized) momentum." (Tabor, 1988, p. 44).

In such a case:

\[ L = \left( \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 \right) + \sum_{j=1}^{N} \frac{1}{2} m_j \dot{q}_j^2 \]

i.e., \( L \) = kinetic energy, which is denoted by \( T \).

When interaction among particles is considered, the potential energy should be considered, and some force law opposes, contained in the potential energy. Referring to Tabor, "the potential energy energy, \( V = V(q_1, q_2, \ldots, q_N) \),"

"and experiences has shown " (according to Landau-Bitslitz) that the correct form of the Lagrangian is:

\[ L \equiv T - V \]

\[ L \equiv \sum_{j=1}^{N} \left( \frac{1}{2} m_j \dot{q}_j^2 - V(q_1, \ldots, q_N, t) \right) \]

Definition of the Lagrangian.
Remark 1. The form of the Lagrangian: Free Particle

Following Landau-Hilfesite, if we require \( L \) to be

(a) homogeneous in time, hence

(b) homogeneous in space (free particle)

(c) isotropy in space

we must impose on \( L \):

(a) to be independent of \( t \) (only implicit depending on \( t \)).

(b) to be independent of \( x \).

(c) do not depend on \( \dot{\vec{v}} \), but only on \( ||\dot{\vec{v}}||^2 = v^2 \).

Hence:

\[ L = L (v^2) \] \hspace{1cm} \text{(XI)}

Now, because of the "principle of least action",
the Lagrangian can differ from other Lagrangians
by the exact definition:

\[ \tilde{L} (q, \dot{q}, t) = L (q, \dot{q}, t) + \frac{df}{dt} \] \hspace{1cm} \text{(XII)}

Thus:

\[ \tilde{S} = \int_0^T \tilde{L} dt = \int_0^T L dt + f(T) - f(0) \]

and the actions only differ from a constant.

\[ = 2 = \text{const} \]
Now, since
\[ f = f(v^2), \]
if we change \( v \) by a small amount \( \epsilon U_0 \)
\[ \tilde{v} = v + \epsilon U_0 \]
we have:
\[ f = f(\tilde{v}^2) = f(v^2 + 2v\epsilon U_0 + \epsilon^2 U_0^2) \]
\[ = f(v^2) + \frac{\partial f}{\partial (v^2)} 2v\epsilon U_0 + O(\epsilon^2). \]
Neglect \( O(\epsilon^2) \). Thus, the second term is a linear function of \( \epsilon \), if \( \frac{\partial f}{\partial (v^2)} \) is a constant:
\[ \frac{\partial f}{\partial v^2} = \frac{1}{2}. \]
Hence:
\[ f(\tilde{v}^2) = \frac{1}{2} m v^2. \]

Now, if two reference frames \( K \) and \( K' \) differ by a velocity \( \tilde{V} / \text{constant} \),
\[ f(\tilde{v}^2) = \frac{1}{2} m \tilde{v}^2 = \frac{1}{2} m (v + \tilde{V})^2, \]
\[ = \frac{1}{2} m v^2 + m v \cdot \tilde{V} + \frac{1}{2} m \tilde{V}^2 \]
\[ = \frac{1}{2} m v^2 + \frac{1}{4} (m \tilde{V} \cdot \tilde{V} + \frac{1}{2} m \tilde{V}^2 t) \quad \text{and} \]
the lost term is \( m \frac{\partial}{\partial v^2} \tilde{\tilde{V}} = \epsilon^2 \), derivative second on being
The last term is an exact derivative, so it can be neglected. Therefore the form of the Lagrangian is

\[ \mathcal{L} = \frac{1}{2} m v^2 \]

And for many particles

\[ \mathcal{L} = \sum_{a=1}^{N} \frac{1}{2} m a v_a^2 \]

Ladam claims that the def'n of the mass should be:

\[ m = \sum \frac{\partial \mathcal{L}}{\partial \dot{q}_a} \]

but he does not explain why.

For the free particle, \( \mathcal{L} \) is independent of \( \dot{q} \). Hence Euler-Lagrange becomes:

\[ -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow \frac{d}{dt} \left( \frac{2\dot{q}}{\dot{q}} \right) = 0 \]

\[ \Rightarrow \frac{\dot{q}}{\dot{q}} = \text{const.} \Rightarrow \frac{\dot{q}}{\dot{q}} = \frac{0(\dot{q})^2}{\dot{q}} = \text{const.} \]

\[ \Rightarrow \dot{q} = \text{const.} \]

\[ \Rightarrow \frac{\dot{q}}{\dot{q}} = \text{const.} \]

If we assume the mass constant \( \Rightarrow \dot{q} = \text{const.} \) and coincides with the idea of a free particle.

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In other words, in an inertial frame, a particle moves with constant velocity:

\[ \mathbf{v} = \text{const.} \]

that moves at the same rate and in the same direction. This is the law of inertia.

End of the Remark.

Remark 2.
There is still the question of how the long-range laws the form:

\[ f = T - U, \]

Tolman claims that London gives an argument, but I cannot find such argument in London. - LifeLite Vol I.

End of Remark 2.
The force is given by:

\[ F_j = -\frac{2}{\partial q_j^2} V(q_1, q_2, \ldots, q_n) \]

and Newton's law by (equ (1.3)):

\[ \frac{d}{dt} (m_j q_j) = -\frac{2}{\partial q_j} V \]

If \( m_j \) are constants:

\[ m_j \frac{d^2 x_j}{dt^2} = -\frac{2}{\partial q_j} V \]

2.1.6 Properties of Lagrangean.

Proposition 1: If the Lagrangean does not depend on time, i.e., \( \frac{\partial L}{\partial t} = 0 \), then

\[ L - \sum_{k} \frac{\partial L}{\partial q_k} q_k = \text{constant} \]

Proof: Let us compute the derivative (in time) of the Lagrangean:

\[ \frac{d}{dt} L(t, q, \dot{q}) = \frac{\partial L}{\partial t} + \sum_{k} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \]

By Euler-Lagrange equations,

\[ \frac{\partial L}{\partial t} + \sum_{k} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = 0 \]

= 0
Hence:
\[
\frac{d}{dt}\left(\mathbf{r} - \sum_k \frac{m_k \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{r}_k\right) = \frac{\partial \mathbf{r}}{\partial t}
\]

Since \( \mathbf{r} \) is \( t \)-independent, \( \frac{\partial \mathbf{r}}{\partial t} = 0 \), so that

\[
\mathbf{r} - \sum_k \frac{m_k \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \mathbf{r}_k = \text{const.} \tag{\star}
\]

Q.E.D.

Now, we proceed to describe (\star).

If \( q = x \) are the cartesian coordinates, notice that

\[
\mathbf{L} = \sum_{j=1}^{\infty} \frac{1}{2} m_j \mathbf{x}_j \times \mathbf{x}_j - \nabla (x_1 \cdots x_n)
\]

Hence

\[
\frac{\partial \mathbf{L}}{\partial \mathbf{x}_k} = m_k \mathbf{x}_k
\]

which is the momentum in the \( x_k \)-direction.

Therefore, we define the generalized momentum:

\[
\mathbf{P}_k = \frac{\partial \mathbf{L}}{\partial \mathbf{r}_k}
\]

and the generalized force:

\[
\mathbf{F}_k = \frac{d}{dt} \mathbf{P}_k
\]

\( = 0 \)
which is by the definition of generalized momentum:

\[ F_k = \frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \right) \]

and, by Euler-Lagrange eqns:

\[ F_k = \frac{\partial L}{\partial q_k} \]

which be considered as our rigorous definition of generalized force.

Notice then, that:

\[ P_k = \frac{\partial H}{\partial \dot{q}_k} = m_k \ddot{q}_k \]

So that: eqn (1.4) is:

\[ \mathbf{L} = \sum_k m_k \ddot{q}_k = \text{const.} \quad \cdots \quad (\star) \]

Using the defn of Lagrange's eqn \((2.\star)\):

\[ \sum_k \frac{1}{2} m_k \dot{q}_k^2 - V(q_1, \ldots, q_n) = \sum_k m_k \dot{q}_k^2 = \text{const.} \]

\[ \text{i.e.} \quad \sum_k \frac{1}{2} m_k \dot{q}_k^2 - V(q_1, \ldots, q_n) = \text{const.} \quad (\star \star) \]

Define the energy as:

\[ E = \sum_{k=1}^n \frac{1}{2} m_k \dot{q}_k^2 + V(q_1, \ldots, q_n) = \text{const.} \quad (\star \star \star) \]

\[ = 0 \]
We then see that eqn. (4.0.0) implies (5.0.0), and so (5.0.0) which means:

\[ -E = \text{const.} \]

i.e., the total energy is constant.

Proposition 2 can be restated as follows:

**Proposition 2.** Let us define:

\[ E = \frac{1}{2} \sum_{k=1}^{n} m_k \dot{q}_k^2 + V(q_1, \ldots, q_n) \]

be the total energy of a system. Then, if the Lagrangian:

\[ \mathcal{L}(t, \dot{q}, q) = \frac{1}{2} \sum_{k=1}^{n} m_k \dot{q}_k^2 - V(q_1, \ldots, q_n) \]

is explicitly independent of time, i.e., if

\[ \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0, \]

then

\[ E = \text{const.} \]

**Proposition 2.** Assume \( \mathcal{L} \) is independent of \( q_k \).

Then \( p_k = \text{const.} \)

**Proof:** From the Euler-Lagrange equations:

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) = \frac{\partial \mathcal{L}}{\partial q_k} \]

\[ = 0 \]
Since \( r \) does not depend on \( T_k \):

\[
\frac{\partial r}{\partial q_k} = 0,
\]

hence

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = 0
\]

i.e.,

\[
\frac{\partial r}{\partial q_k} = \text{const}
\]

and by the definition of generalized moment (equ. (1.32)):

\[
P_k = \text{const}.
\]

QED.

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**Definition.** If \( q_k \) is a coordinate such that

\[
\frac{\partial r}{\partial q_k} = 0
\]

(i.e., \( P_k \) is a missing coordinate), then

\( q_k \) is called a cyclic coordinate.

---

By use of the generalized momentum, \( P_k = \frac{\partial H}{\partial q_k} \), the energy can be written as:

\[
E = -\left( P - \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \right)
\]

(by comparing (5.33) to (4.13)). And by the definition of the generalized momentum:

\[
E = \sum_{k=1}^{n} P_k \dot{q}_k - L(t)
\]
**Proposition 3**: If a system is closed, and the space is homogeneous, then the total momentum is constant:

\[ \sum_{k=1}^{n} P_k = \text{const} \]

**Proof**: If the system is homogeneous and closed, and the space is homogeneous, then the total effect of all the forces should be zero:

\[ \sum_{j=1}^{2} F_j = 0 \]

Since the generalized force is:

\[ F_j \equiv \frac{\partial L}{\partial \dot{q}_j} - \frac{d}{dt} \frac{\partial L}{\partial q_j} = \frac{d}{dt} P_j \]

Euler-Lagrange

thus:

\[ \sum_{j=1}^{2} \frac{d}{dt} P_j = 0 \Rightarrow \sum_{j=1}^{2} P_j = \text{const} \]

(O.D.E.)

(This is Newton's Third Law).

Homogeneity in time \( \Rightarrow \) Conservation of energy

Homogeneity in space (and closed) \( \Rightarrow \) Conservation of momentum

Noether's theorem

Lagrangian Mechanics:

Generalized coordinates $q_i$ and generalized velocities are considered independent variables.

Now, generalized momentum, $p_i = \frac{\partial L}{\partial \dot{q}_i}$, are introduced and they are considered as independent variables from $q_i$ and $\dot{q}_i$.

V. Arnold Hamiltonian mechanics: geometry in phase space.

Proposition 4. The momentum is the gradient of a scalar field.

Proof: Assume $q = q(t)$ is a trajectory that is a critical point of the action integral

$$S = \int_{t_1}^{t_2} L(t, q, \dot{q}) \, dt.$$ 

Assume that the trajectory $q = q(t)$ has a fixed end-point at $t = t_1$, but a variable end-point at $t = t_2$. 

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This is to say: the variation of \( q(t) \), given by
\[ \dot{q}(t) = q(t) + h(t) \]
is such that:
\[ q(t_1) = q(t) \quad \text{i.e.} \quad \dot{h}(t_1) = 0 \]
and
\[ q(t_2) = q(t_2) + h(t_2), \quad \text{with} \quad \dot{h}(t_2) = 0. \]

Thus, the variation of the action is:
\[ S_S = S[q + h] - S[q] = \int_{t_1}^{t_2} \left[ L(q, \dot{q}, t) - L(q, \dot{q}) \right] dt \]

three
\[ S_S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial \dot{q}} \dot{h} + \frac{\partial L}{\partial q} \right) dt + O(h^2) \]

and integrating by parts:
\[ S_S = \left( \frac{\partial L}{\partial \dot{q}} \right) h \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) dt + O(h^2) \]

Since \( \dot{h}(t_1) = 0 \) and \( q \) is an actual trajectory (i.e., Euler-Lagrange equations hold) then the integral is zero and so:
\[ S_S = \frac{\partial L}{\partial \dot{q}} h(t_2) + O(h^2) \]

For a system of \( n \) generalized coordinates:
\[ SS = \sum_{k=1}^{n} \frac{\partial}{\partial k} \left[ \frac{\partial}{\partial k} \right]^{(k)} \left( h_2 \right) + O(h^7) \]

so that by the definition of generalized momentum: \( P_k = \frac{\partial}{\partial k} \), hence:

\[ SS = P \cdot h(1) + O(h^7) \]

or

\[ SS = \sum_{k=1}^{n} P_k \cdot h_k (1) + O(h^7) \]

Therefore, the moments can be written as:

\[ P_k = \frac{\partial S}{\partial q_k} \]

at a given time along a given extremal path.

\[ \text{[if we thought as } h_k(t) = S q_k(t) \implies SS = p \Delta q \]

\[ \implies \frac{SS}{\Delta q} = P \quad \text{or: } \quad SS = \sum P_k \Delta q_k \implies \]

\[ \frac{dS}{dt} = \sum P_k \delta t_k \implies \frac{SS}{\Delta q_k} = P_k \]

and the moments are the gradients of the forces.

\[ \text{GREAT!} \]

\[ \frac{\text{GREAT}}{?} \]

\[ \gamma = \sqrt{1 - \frac{1}{2}} = 1 \]
2.2. Hamiltonian Formulation of Mechanics

1. Mechanics description in the \((q, p)\) variables.
2. Contains the same info as in Lagrangian Mechanics.
3. It is the natural setting for Quantum and Statistical Mechanics extending.
4. It is also rich in Perturbation theory.
5. Hamiltonian phase-space is the appropriate space to describe what Integrability and
   non-integrability are, and to describe chaotic behavior in non-integrable systems.

According to H. Fleschka's notes:

1. Problems in Hamiltonian Mechanics.
   1. Decide if a system is integrable or not.
      If possible, find its Hamilton integrals.
   2. Are the solutions of the Hamilton equations bounded for all \(t\)? For all \(t\) do the solutions
      of the Hamilton equations exist for all \(t\)?
      If possible find its solutions
   3. Understand the geometry of the phase plane:
      (a) Qualitative properties of the motion
      (b) Quantitative properties of the motion
      (c) Why the system possesses the
      related symmetries. (A symmetry corresponds to a constant of \(\mathcal{F} = 1\); motion: Noether's theorem).
Problem 3 is of special interest, since it brings together several mathematical ideas.

This was according to Fleschke. According to James Meiss, in the Encyclopedia of Nuclear Science (2004; A. Scott, Editor), it is as follows.

**BENEFITS OF USING HAMILTONIAN FORMULATION.**

2. Henri's Poincaré's geometric formulation:
   - Symplectic Geometry.
3. The concept of integrability is expressed.
4. A nice way to express Probabilistic Theory.
5. Nearly-integrable Hamiltonian systems provide a remarkable stability expressed by KAM Theory (Kolmogorov–Arnold–Moser).
6. Hamiltonian formalism provides the natural formulation for Quantum and Statistical Mechanics.

**INTEGRABILITY**

Vague definition. A system of differential equations that can be explicitly solved for arbitrarily initial conditions is said to be completely integrable. (Zeldovich, 1991. What is integrability?)
2.2.2 Transformation to the Hamilton Picture

The idea is to move from the Lagrangian picture, in which the independent variables are \((q, \dot{q})\), to the Hamiltonian picture, in which the phase-space picture in which the independent variables are \((q, p)\). The time \(t\) works as a parameter.

Given the Lagrangian:

\[ L = L(t, q, \dot{q}) \]

where \( q = (q_1, q_2, \ldots, q_n) \), and \( \dot{q} = \frac{dq}{dt} \),

the conjugate variable (the generalized momentum) can be defined as:

\[ \pi_j = \frac{\partial L}{\partial \dot{q}_j} \]

If

\[ \det \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right) \neq 0 \]

then, equation \((\star)\) can be inverted for \( \dot{q}_i \) and expressed \( \dot{q}_i \) as a function of \((t, q, p)\).
Keeping this in mind, we defined the Hamiltonian:

\[ H(t, q, p) = \left( \sum_j \frac{i_j p_j}{2} \right) - L(t, q, \dot{q}) \]

(since, on the RHS we can express \( \dot{q}_j = \frac{i_j}{p_j} (t, q, p) \)

Example: Motivation

We know:

\[ L = \left( \sum_{j=1}^n \frac{1}{2} \left( m_j \dot{q}_j \right)^2 \right) - V(q_1, ..., q_n) \]

We determine:

\[ p_j = \frac{\partial L}{\partial \dot{q}_j} = m_j \dot{q}_j \]

Since \( \frac{\partial^2 L}{\partial q_j^2} = m_j \neq 0 \),

\[ \Rightarrow \quad \dot{q}_j = \frac{1}{m_j} p_j \]

So, that:

\[ H = \left( \sum_j \frac{\dot{q}_j p_j}{2} \right) - L(t, q, \dot{q}) \]

\[ = \sum_j \frac{1}{m_j} p_j \dot{p}_j - \left( \sum_{j=1}^n \frac{1}{2} m_j \dot{q}_j^2 \right) + V(q_1, ..., q_n) \]

\[ = \sum_j \frac{1}{m_j} p_j \dot{p}_j - \sum_j \frac{1}{2} m_j \frac{\dot{q}_j^2}{m_j^2} + V(q_1, ..., q_n) \]

\[ = \sum_j \frac{1}{m_j} p_j \dot{p}_j - \sum_j \frac{1}{2} \frac{\dot{q}_j^2}{m_j} + V(q_1, ..., q_n) \]
\[ H = \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V(q_1, \ldots, q_N) \]

and the Hamiltonian coincides with the usual expression of the energy, \( E = T + V \).

**Example 2.** Consider a bead of mass moving on a frictionless wire of shape:

\[ y = f(x) \]

\[ T = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) \]

But \[ y = f(x) \Rightarrow \frac{dy}{dt} = f'(x) \frac{dx}{dt} \]

They then kinetic energy under the constraint is:

\[ T = \frac{1}{2} m \left( \dot{x}^2 + (f'(x))^2 \dot{x}^2 \right) = \frac{1}{2} m \dot{x}^2 (1 + (f'(x))^2) \]
The potential energy is simply:
\[ V = mgh = mg y = mg f(x) \]
Then, the Lagrangean is:
\[ L = \frac{1}{2} m \dot{x}^2 \left( 1 + \left( f'(x) \right)^2 \right) - mg f(x) \]
The generalized momentum is:
\[ p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \left( 1 + \left( f'(x) \right)^2 \right) \]
and so:
\[ \dot{x} = \frac{p}{m \left( 1 + \left( f'(x) \right)^2 \right)} \]
and the Hamiltonian is:
\[ H(p, x) = x p - L = \frac{p^2}{m \left( 1 + \left( f'(x) \right)^2 \right)} - \left( \frac{m \dot{x}^2 \left( 1 + \left( f'(x) \right)^2 \right)}{2} - mg f(x) \right) \]

\[ H(p, x) = \frac{p^2}{2 m \left( 1 + \left( f'(x) \right)^2 \right)} \left( 1 + \left( f'(x) \right)^2 \right) + mg f(x) \]

\[ H(p, x) = \frac{p^2}{2 m \left( 1 + \left( f'(x) \right)^2 \right)} + mg f(x) \]
**Equation:**

\[ \frac{dH}{ds} = \sum i \frac{dP}{ds} - \sum \frac{\partial H}{\partial q_i} \frac{dq_i}{ds} \]

**Note:**

As can be seen when calculating the derivative of

\[ H(t(s), q(s), p(s)) \]

\[ \frac{dH}{ds} = \sum \left( \frac{\partial H}{\partial q_i} \frac{dq_i}{ds} + \frac{\partial H}{\partial p_i} \frac{dp_i}{ds} \right) \]

**Equation:**

\[ \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} \]

\[ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \]

**Note:**

Combining \((*)\) and \((***)\):

\[ \frac{dq_i}{ds} = \frac{\partial H}{\partial p_i} \]

\[ \frac{dp_i}{ds} = \frac{\partial H}{\partial q_i} \]

**Note:**

Now, for the generalized force \( F_i = P_i \)

\[ \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} \]

\[ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \]

By definition of generalized force (250).

**Note:**

And substituting in \((***)\), we then obtain:

\[ \frac{dq_i}{dr} = \frac{\partial H}{\partial p_i} \]

\[ \frac{dp_i}{dr} = -\frac{\partial H}{\partial q_i} \]

Newton's equations of motion (\(\ast\))

\[ \frac{dH}{dt} = \frac{\partial H}{\partial q} \]

\[ \frac{dH}{dt} = \frac{\partial H}{\partial p} \]

\[ \frac{1}{2} v^2 = \frac{dH}{dt} \]
The previous equ means that if H does not depend explicitly on t, it also does not depend on t.

But previously (Proposition 1), we showed that if \( \frac{\partial H}{\partial t} = 0 \), then the energy \( E = \text{const} \).

Hence, if \( \frac{\partial H}{\partial t} = 0 \), the energy \( E = \text{const} \).

Now, from equ (19.11), e of the parameter \( s \) is the time \( t \); \( t = s \). Then:

\[
\frac{dH}{dt} = \sum_i \left( 2H \frac{dq_i}{dt} + 2H \frac{dp_i}{dt} \right) + \frac{\partial H}{\partial t}.
\]

Using Hamilton's equations, (19.11):

\[
\frac{dH}{dt} = \sum_i \left[ 2H \frac{\partial q_i}{\partial p_i} + \frac{\partial H}{\partial p_i} \right] - \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t}.
\]

i.e.

\[
\frac{dH}{dt} = \frac{dH}{dt}.
\]

So, if \( \frac{\partial H}{\partial t} = 0 \) i.e., H does not explicitly depend on t, then \( \frac{dH}{dt} = 0 \), i.e., H is a constant of motion.
As a fundamental part in the integration of the equations of motion, we require to identify integrals of motion. One of them is the Hamiltonian if \( \frac{dH}{dt} = 0 \).

Consider a quantity that depends on \( q, p, t \):

\[ f = f(q, p, t). \]

Thus:

\[
\frac{df}{dt} = \sum \left( \frac{\partial f}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} \right) + \frac{df}{dt}
\]

\[
= \sum \left( \frac{\partial f}{\partial q} \frac{dq}{dt} - \frac{\partial f}{\partial p} \frac{dp}{dt} \right) + \frac{df}{dt}
\]

Define the Poisson bracket:

\[ \{f, H\} = \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \]

Thus:

\[ \frac{df}{dt} = \{f, H\} + \frac{df}{dt}. \]

**Theorem:** (Poisson's Theorem).

\( f \) is an integral of motion if

\[ \{f, H\} = 0 \quad \text{and} \quad \frac{df}{dt} = 0. \]
What we did on page 20 was the following. Notice that

\[ [H, H] = 0 \]

Hence, if \( \frac{dH}{dt} = 0 \) \( \Rightarrow \frac{dH}{dt} = 0 \)
and \( H \) is an integral of motion.

Then the Poisson bracket:

\[ [f, g] = \frac{1}{2} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \]

for \( f, g, h \in \mathcal{C}^1 \) (functions of \( t, p, q \)) has the following properties:

\[ [f, g] = -[g, f] \]

\[ [f + g, h] = [f, h] + [g, h] \]

\[ [fg, h] = \frac{d}{dt} \left( h[q, f] \right) \]

and

\[ [f, [g, h]] + [g, [f, h]] + [h, [f, g]] = 0 \]

This is called the Jacobi's identity.
Notice that the Poisson bracket "satisfy" what is called a Lie algebra. (I am not sure on this assertion. The Poisson bracket is not a set of elements which constitutes an algebra. To begin with, the Poisson bracket is not a set of elements). (I think that Dr. Tabor means that the functions $f, g, h$ (set of functions) satisfy the Lie algebra).

Notice that:

\[ [p_i, q_j] = 0, \]
\[ [p_i, p_j] = 0, \]
\[ [q_i, q_j] = 0, \]

Assume $f, g$ to be three independent.

Claim: If $f$ and $g$ are constraints of motion, then $[f, g]$ is also a constraint of motion.

Proof: \[ \frac{df}{dt} = 0, \quad \frac{dg}{dt} = 0 \quad \Rightarrow \quad [f, H] = 0, \quad [g, H] = 0 \]

Substitute these expressions into Jacobi's identity:

\[ 0 + 0 + [H, [E, H]] = 0 \quad \Rightarrow \quad \frac{d}{dt} \frac{[E, H]}{[E, H]} = 0 \quad \forall E \in \mathbb{R} \]
The flux given by Hamilton's equation:

\[
\begin{align*}
\psi_i &= \frac{\partial H}{\partial p_i} = \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) \\
P_i &= -\frac{\partial H}{\partial q_i} = \frac{\partial}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right)
\end{align*}
\]

is incompressible, i.e., it is divergence-free.

Proof: Notice that Hamilton's equations (eqn (\text{x})) is a vector field:

\[
\begin{align*}
\dot{q}_i &= f_i (q, p, t) \\
\dot{p}_i &= g_i (q, p, t)
\end{align*}
\]

Then, \((f, g) = (f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n)\) is a flow (vector field) and, as we will see, \(\psi\) is a divergence-free field. Compute

\[
\text{div}(\mathbf{f}) = \frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \cdots + \frac{\partial f_n}{\partial q_n} + \frac{\partial g_1}{\partial p_1} + \frac{\partial g_2}{\partial p_2} + \cdots + \frac{\partial g_n}{\partial p_n}
\]

\[
= \sum_{i=1}^{n} \left( \frac{\partial f_i}{\partial q_i} + \frac{\partial g_i}{\partial p_i} \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{\partial}{\partial q_i} \left( \frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) \right)
\]

\[
= 0 \text{ if } H \in C^2
\]

\[\triangledown \psi \]
2.2.6 Hamilton's equation of motion.

Consider the Hamiltonian, as defined in eqn (5.1*).

\[ H(t, q, p) = \sum_j q_j P_j - L(t, q, \dot{q}) \]

Remember that:

\[ P_j = \frac{\partial L}{\partial \dot{q}_j} \]

and if \( \det \begin{vmatrix} \dot{q}_j \\ \dot{q}_i \end{vmatrix} \neq 0 \), we can invert the previous eqn to get \( \dot{q}_i \) as a function of \( t, q, p \).

\[ \dot{q}_i = \frac{\partial f_i}{\partial q_i} \left( t, q, p \right) \]

Then: Assume new \( t, q \) and \( p \) are functions of some parameter \( s \), \( t = t(s) \), \( q = q(s) \), \( p = p(s) \).

Overall, the Hamiltonian will be a function of \( s \):

\[ H(s) = H(t(s), q(s), p(s)) \]

If we require to compute \( \frac{dH}{ds} \) using (7) above, we have:

\[ \frac{dH}{ds} = \sum_i \frac{d}{ds} \left( P_i \dot{q}_i \right) - \frac{d}{ds} \left( L(t, q, \dot{q}) \right) \]

\[ = \sum_i \frac{dP_i}{ds} \dot{q}_i + P_i \frac{d}{ds} \left( \dot{q}_i \right) - \left[ \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{ds} + \frac{\partial L}{\partial p_i} \frac{dp_i}{ds} \right] \]

by definition of \( P_i = \frac{\partial L}{\partial \dot{q}_i} \):

\[ = \sum_i \frac{dP_i}{ds} \dot{q}_i + P_i \frac{d}{ds} \left( \dot{q}_i \right) - \left[ \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{ds} + \frac{\partial L}{\partial p_i} \frac{dp_i}{ds} \right] \]

\[ = \sum_i \frac{dP_i}{ds} \dot{q}_i + P_i \frac{d^2}{ds^2} \left( \dot{q}_i \right) - \left[ \sum_i \frac{\partial L}{\partial q_i} \frac{dq_i}{ds} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{ds} + \frac{\partial L}{\partial p_i} \frac{dp_i}{ds} \right] \]
This is basically Liouville's Theorem.

Hamiltonian flows preserve volumes in phase space.

$$\int dq \, dp = \int \frac{\Theta(q, p)}{\Theta(q_0, p_0)} \, dq_0 \, dp_0,$$

where $J_0$ is a region at some initial subset in the phase space; and $J_2$ is the region evolved as the system moved from $t=0$ to $t>0$.

For a system of $n$ ODE's, we require $n$ constraints of integration (Tabor claims that this is true, but only one is required). For a Hamiltonian system in $n$-dimensional physical space, i.e., a $2n$-dimensional phase space, we have $2n$ ODE's. Surprisingly, we only require $n$ integrals of motion to describe the motion: no more, no less.

To see how this works, let's go to Fresnel's notes on Hamiltonian systems.
Integrability (Floshko's rule).
Assume that the Hamiltonian system.

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}
\end{align*}
\]

has \( k \) constants of motion:

\[
F_i(t, q, p) = C_i
\]

\[
F_k(t, q, p) = C_k
\]

and they satisfy:

\[
\left[ F_i, F_j \right] = 0, \quad \forall i, j = 1, 2, \ldots, k.
\]

Lemma: If \( \forall F_1, F_2, \ldots, F_k \) are linearly independent at some point \((q_0, p_0)\) in the phase-space, then \( k \leq n \).

Corollary: There exists, at most, \( n \) independent functions in evolution on \( \mathbb{R}^n \).

Proof of the lemma:
Take \( C = C_0 \) a constant and \((q_0, p_0) \in F_0 = \left\{ F(q, p) \in \mathbb{R}^n \mid F(q, p) = C_0 \right\} \)

\[
= \left\{ F(q, p) \in \mathbb{R}^n \mid F_1(q, p) = C_1, \ldots, F_k(q, p) = C_k \right\}
\]

where \( C = C_0 = (C_1^{(0)}, \ldots, C_k^{(0)}) \)

\[
= 26 =
\]
We have that
\[ \dim F_c = 2n - k, \text{ by the Implicit Function Theorem.} \]

The vectors
\[
\left( \frac{\partial F_1}{\partial p_1}, \ldots, \frac{\partial F_1}{\partial p_n} \right), \ldots, \left( \frac{\partial F_k}{\partial q_1}, \ldots, \frac{\partial F_k}{\partial q_n} \right)
\]
are linearly independent and tangent to \( F_c \).

This implies that:
\[ \dim F_c \geq k, \text{ since } F_c \text{ has } k \text{ tangent vectors (but could have more).} \]

Therefore,
\[ 2n - k \geq k \]
\[ \Rightarrow n \geq k. \]

\( \square \)

Thus, the number \( k \) of integrals of motion cannot be greater than the dimension \( n \) of the physical space.

(Or, \( k \) cannot be greater than half of the dimension of the phase space: \( \frac{2n}{2} = n \).)
2.3 Canonical Transformations.

In the Lagrangian description, it is convenient to change from \( q \) to \( \theta \):

\[
\theta = \theta(q_1, q_2, \ldots, q_n) \quad (x)
\]

(say, from Cartesian to polar coordinates).

In the Hamiltonian point of view, \( q \) and \( p \) are considered as independent coordinates. Then, we expect to change coordinates

\[
(q, p) \rightarrow (\theta, P),
\]

i.e.,

\[
\theta = \theta(q_1, \ldots, q_n, p_1, \ldots, p_n) \quad (x)
\]

\[
p_i = p_i(q_1, \ldots, q_n, p_1, \ldots, p_n) \quad (x)
\]

Definition: \( \theta = \theta(q_1, \ldots, q_n) \)

and \( P = P(p_1, p_2, \ldots, p_n) \).

These transformations are called point transformations.

Definition: \( \theta = \theta(q, p) \) and \( P = P(q, p) \) as in \((x)\) above. They are called canonical transformations when the symplectic structure is preserved, i.e.,

\[
\frac{\partial \theta}{\partial P_i} = \frac{\partial H}{\partial p_i}
\]

\[
\frac{\partial p_i}{\partial P_i} = 2\pi
\]
Where \( \hat{H}(0,p) = H(q(0,p), p(0,p)) \), and then \( H(q,p) \) is the Hamiltonian in the \( q,p \)-variables. Similarly \( H(q,p) = \hat{H}(q(0,p), p(0,p)) \).

### Section 2.3.3. The Preservation of Phase-Space Volume

**Theorem.** A fundamental property of canonical transformations is that phase-space volume is preserved:

\[
\int \prod_{i=1}^{n} dq_i dp_i = \int \prod_{i=1}^{n} dq_i dp_i \tag{\star}
\]

where \( S_2 \subset \{(q,p) \mid q \in \mathbb{R}^n, p \in \mathbb{T}^n \} \).

**Proof.** Using the change of variables

\[
\begin{align*}
\Theta &= \Theta(q,p), \\
P &= P(q,p)
\end{align*} \tag{\star}
\]

Then:

\[
\int_{S_2} dp d\omega = \int_{S_2} P(q,p) dq dp
\]

where \( P(q,p) \) is the Jacobian of the \( \Theta(q,p) \) transformation (\star).

If \( \frac{\partial (0,p)}{\partial (q,p)} = 1 \), then (\star) holds.
Remark: Now that I have stated and proved the previous "theorem", it is not clear to me why a canonical transformation preserves volume.

I.e., we require to prove

$$\mathcal{Q} = \mathcal{Q}(q, p), \quad \mathcal{P} = \mathcal{P}(q, p), \quad \left\{ \begin{array}{c}
\mathcal{Q}_q = \frac{\partial \mathcal{Q}}{\partial q}, \\
\mathcal{P}_p = -\frac{\partial \mathcal{Q}}{\partial p}
\end{array} \right\} \Rightarrow \left( \frac{d\mathcal{Q}}{dq} \right) \left( \frac{d\mathcal{P}}{dp} \right) = 1,$$

and so

$$\int dq \, dp = \int dq \, d\mathcal{P}.$$  

It is required to prove $$(\star \star)$$.

**Example 1:**

$$\mathcal{Q} = -P,$$

$$\mathcal{P} = q.$$

$$\left( \frac{d\mathcal{Q}}{dq} \right) \left( \frac{d\mathcal{P}}{dp} \right) = \det \begin{pmatrix} \mathcal{Q}_q & \mathcal{Q}_p \\ \mathcal{P}_q & \mathcal{P}_p \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1.$$

Then, $(\star \star)$ is a volume preserving (canonical) transformation.
Example 2: Polar to cartesian coordinates:

\[ q = P \cos \theta \]
\[ p = P \sin \theta . \]

Where \((q, p)\) are the cartesian coordinates, and

Thus:

\[ \frac{\partial (q, p)}{\partial (Q, P)} = \det \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix} = -\det \begin{pmatrix} -P \sin \theta & -P \cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} \]

\[ = -P \sin^2 \theta - P \cos^2 \theta = -P. \]

Then, this transformation does not preserve phase-space volumes.

Hao's Theorem:

Volume in phase space is preserved under Hamiltonian flows.

Proof: Consider \( q = q(t) \) \( p = p(t) \) a trajectory in the phase space. Say \( q_0 = q(0) \) and \( q_1 = q(t) \)

Thus:

\[ P_0 = p(0) \]
\[ P_1 = p(t). \]
\[ q_1 = q(t_1) = q(0 + t_1) = q(0) + t_1 \frac{\partial q}{\partial t} + O(t_1^2) \]

\[ = q_0 + t_1 \frac{\partial H}{\partial \phi}(q_0, p_0, 0) + O(t_1^2). \]

\[ p_1 = p(t_1) = p(0 + t_1) = p(0) + t_1 \frac{\partial p}{\partial t} + O(t_1^2) \]

\[ = p(0) + t_1 \left( -\frac{\partial H}{\partial \theta}(q_0, p_0, 0) \right) + O(t_1^2). \]

The transformation \((q_0, p_0) \rightarrow (q_1, p_1)\), given by the Hamiltonian flow, is canonical; indeed:

\[
\frac{\partial (q_1, p_1)}{\partial (q_0, p_0)} = \det \begin{pmatrix}
\frac{\partial q_1}{\partial q_0} & \frac{\partial q_1}{\partial p_0} \\
\frac{\partial p_1}{\partial q_0} & \frac{\partial p_1}{\partial p_0}
\end{pmatrix} = \det \begin{pmatrix}
1 + t_1 \frac{\partial^2 H}{\partial \theta^2} & -t_1 \frac{\partial^2 H}{\partial \theta \phi} \\
-t_1 \frac{\partial^2 H}{\partial \theta \phi} & 1 - t_1 \frac{\partial^2 H}{\partial \phi^2}
\end{pmatrix}
\]

\[= 1 + t_1 \left( \frac{\partial^2 H}{\partial \phi^2} - \frac{\partial^2 H}{\partial \theta \phi} \frac{\partial^2 H}{\partial \phi^2} \right) + t_1^2 \frac{\partial^2 H}{\partial \theta^2} \frac{\partial^2 H}{\partial \phi^2} + O(t_1^2). \]

Since here \( p = p_0, q = q_0 \), the above form vanishes:

\[ \frac{\partial (q_1, p_1)}{\partial (q_0, p_0)} = 1 + O(t_1^2) \]
Then, over any finite period of time $[0, t_1]$, the change of the volume in phase-space is of the order $O(t_1^2)/t_1 = O(t_1)$ as $t_1 \to 0$.

The phase-space volume is then preserved under a Hamiltonian flow, which is an infinitesimal transform.

Then,

$$\frac{\partial (q_1, p_1)}{\partial (q_0, p_0)} \to 1$$

and the phase-space volume in the $(q_0, p_0)$ is preserved under the transformation $(q_0, p_0) \to (q_1, p_1)$ (under the Hamiltonian flow). This is Liouville's theorem statement. Q.E.D.

**Rank:** In any case, $(q_0, p_0) \to (q_1, p_1)$ preserves volumes in phase space, but Larmor does not prove that:

$$\frac{p_1}{q_1} = 2\pi i (q_1, p_1)$$

This is a requirement to be a $\mathbb{Z}_3$ canonical transformation.
Section 23.6 The Optimal Transformation

This is a key section, since here it is introduced the concept of integrations of the Hamilton equation under constant coordinates, i.e., in the action-angle variables without saying they are so.

We wish to find a transformation, canonical, and

\[ \theta = \theta(q, p) \quad \text{invariant}, \]
\[ p = p(q, p) \]

such that:

\[ H(q, p_1, ..., p_n) \] \( \rightarrow \) \( H(p_1, p_2, ..., p_n) \)

(1)

\[ H(p_1, p_2, ..., p_n) = H(q(q_0, p), p(q_0, p)) \]

i.e. \( H \) depends on \( p \) only. Since the transformation is canonical,

\[ \dot{\theta} = \frac{\partial H}{\partial p} \quad \text{i.e., } \dot{\theta} = \frac{\partial H}{\partial p} \]
\[ \dot{p} = -\frac{\partial H}{\partial q} \]
\[ \dot{p} = -0 \]

i.e., \( P(t) = P_0 \) is a constant, and

and so:

\[ \dot{\theta} = \frac{\partial H(P_0)}{\partial p} \]

\[ = 34 \]
\[ \dot{\theta} = f_o \quad \Rightarrow \quad \dot{\theta}(t) = f_0 t + \Theta_0 \]

\[ P(t) = P_0. \]

i.e.

\[ \Theta_i(t) = \int f_i^{(0)} t + \Theta_i(0). \]

\[ P_t(t) = P_{i_0}(0). \]

\[ \text{here.} \quad \int f_i^{(0)} = \frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial P_i} (P_i(t), P_{i_0}, \ldots, P_{i_{i_0}}) \]

i.e.

\[ f_i = f_i^{(0)} = f_i^{(0)} (P_i(t), \ldots, P_{i_{i_0}}). \]

The new moment, \( P = P(\Theta) \), are constants of motion. 
Actually, \( P(t) = P(\Theta) \) are the integrals of motion ("non-trivial" constants of motion).

\[ \text{and we have } n \text{ of them.} \]

We also require \( n \) "trivial" constants of motion, and they are \( \Theta_i(0) \) \( (i = 1, 2, \ldots, n) \) which are the initial conditions of the system.

We thus can return to the original variables

\[ q(t) = q(\Theta(t), P(t)) \]

which are solutions 

\[ p(t) = p(\Theta(t), P(t)) \]

to the original "trivial" eqn's.
To integrate the equations, Hamilton's equations, we require:

1. To find this magadal transformation
   (to find these magadal coordinates)
2. To know how to correctly transform the
   Hamiltonian into its new representation

Question: Up to this point, it is not clear that a
 canonical transformation:

\[ (q, p) \rightarrow (Q, P) \]

i.e.,

\[ Q = Q(q, p) \]
\[ P = P(q, p). \]

with

\[ Q = \frac{\partial H}{\partial p} \]
\[ P = -\frac{\partial H}{\partial q} \]

preserves volume:

\[ \frac{\partial (Q, P)}{\partial (q, p)} = 1 \]
23. Generating functions

Canonical transformations are reached by relations called generating functions.

Generating functions (as Tabor notes claims) are extremely useful. They provide a way to find the functional relation in between $P$ and $Q$ with the original variables $p$ and $q$. Tabor says this is “two for the price of one”. If one does not exploit the generating function technique, and just starts with a transformation $Q = Q(q, p)$; the other function relation $P = P(q, p)$ will be not an easy task to perform.

There, we will use the preservation of volume to obtain the generating function.

Tabor considers here the one-degree of freedom. Let's consider the transformation:

$$(q, p) \rightarrow (Q, P).$$

Then, assuming this is a canonical transformation, it preserves volume:

$$\int dq dp = \int dQ dP$$

$$= 37.$$
For Stoke's theorem (or the divergence theorem), Gauss Theorem all of them equivalent in $\mathbb{R}^3$, we can write:

$$\oint_{\partial \Omega} \mathbf{F} \cdot d\mathbf{r} = \int_{\Omega} \nabla \cdot \mathbf{F} \, dV$$

or

$$\oint_{\partial \Omega} \mathbf{F} \cdot d\mathbf{r} = \int_{\Omega} \nabla \cdot \mathbf{F} \, dV$$

So we have the transformation:

$$\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{p})$$

$$\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{p})$$

This will be a key identity for this section.

Example: Hence we can invert the 1st eqn of (x); and invert it for $\mathbf{P}$:

$$\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{Q})$$

(this can be done provided: $\frac{\partial \mathbf{Q}}{\partial \mathbf{P}} \neq 0$)

By this implicit function theorem.

So we have the transformation:

$$\mathbf{P} = \mathbf{P}(\mathbf{q}, \mathbf{Q})$$

Thus, eqn (x) becomes:

$$\oint_{\partial \Omega} \mathbf{F} \cdot d\mathbf{r} = \int_{\Omega} \nabla \cdot \mathbf{F} \, dV = 0.$$
We know that, if $F_1 = F_1(q, q)$ is a $C^1$ function on $S^2$, then:
\[ \oint \frac{df_1}{ds} (q, q) ds = 0; \quad (\textit{A}) \]

Noting (\textit{A}) with (\textit{B.8} \times \times), then we can say that there exists an $F_1 = F_1(q, q)$, such that:
\[ \oint \left[ P(q, q) \frac{dq}{ds} - P(q, q) \frac{dF_1}{ds} \right] ds = \oint \frac{df_1}{ds} ds \]
\[ \textit{i.e.} \quad \oint \left[ P(q, q) - \frac{\partial F_1}{\partial q} \right] dq - \left( P(q, q) + \frac{\partial F_1}{\partial q} \right) \frac{dq}{ds} ds = 0 \]
\[ \text{and this is } \forall \Omega \subseteq \mathbb{R}, \text{ arbitrary then:} \]
\[ P(q, q) = \frac{\partial F_1}{\partial q} \quad \quad \textit{(\ast \ast)} \]
\[ P(q, q) = \frac{\partial F_1}{\partial q} \quad \quad \textit{(\ast \ast \ast)} \]

In other words, we have from (\textit{\ast \ast}), that $P$ is a function of $q$ and $0$:
\[ P = P(q, 0) \]
\[ \text{if } \partial F_1 \neq 0; \text{ by the implicit function theorem:} \]
\[ 0 = 0(q, P) \]

Substitute this into (\textit{\ast \ast \ast}), to get:
\[ = 3q = \]
\[ P = P(\alpha, \phi(\alpha, \beta)) = \phi(\alpha, \beta). \]

Then we obtain our canonical transformation

\[ \Omega = \Omega(\alpha, \beta) \]
\[ \Phi = \Phi(\alpha, \beta) \]

**Example 2.** Assume now that we choose \((\alpha, \beta)\) to be our independent variables for the second generating function \(F_2 = F_2(\alpha, \beta).\) (This can be done by assuming \(\Phi = 0\) in (88.1), so that we can invert 2nd eqn in (88.1): \(\Phi = \Phi^{-1}(\alpha, \beta).\)) i.e. \(\alpha = \alpha(\alpha, \beta),\) so we have: \(\Omega = \Omega(\alpha, \beta)\) i.e. \(\Omega = \Omega(\alpha, \beta)\) and \(\Phi = \Phi(\alpha, \beta).\)

We start by observing that:

\[ \Omega = \oint dS \left( \frac{d\alpha}{ds} \right) \Phi = \oint \left( \frac{d\Omega}{ds} + \Phi \frac{d\Phi}{ds} \right) ds \]

\[ \Rightarrow \oint \frac{d\Phi}{ds} ds = - \oint \frac{d\Omega}{ds} ds; \]

Substitute this last eqn into (88.1) to get:

\[ \Omega = \Phi = \text{const}. \]
\[ \oint p \, dq = - \oint O \, dp. \]

and we proceed as in the previous example:

\[ \oint p \, dq + O \, dp = 0. \]

ie

\[ \oint p(q, p) \, dq + O(q, p) \, dp = 0. \]

Since

\[ \oint \frac{dF_2}{ds} \, ds = 0, \text{ if } F_2 \in C^1(\mathbb{R}) \]

there should exist a function \( F_2 = F_2(q, p) \).

and that:

\[ \oint p(q, p) \, dq + O(q, p) \, dp = \oint \frac{dF_2}{ds} \, ds. \]

\[ \oint \left( p(q, p) - \frac{\partial F_2}{\partial q} \right) \, dq + \left( O(q, p) - \frac{\partial F_2}{\partial p} \right) \, dp = 0 \]

\( \forall R \subset \mathbb{R} \), where \( R \) is a region where \( F_2 \in C^1(R) \).

Hence:

\[ p = \frac{\partial F_2}{\partial q}(q, p) \]

\[ O = \frac{\partial F_2}{\partial p}(q, p). \]

Notice that, the 1st eqn is \( p = p(q, p) \), if \( \frac{\partial p}{\partial p} \neq 0; \)

\( \Rightarrow \)
\[ P = p^{-1}(q, P) \text{ i.e. } P = P(q, P). \]
and the 2nd equ in \( \Theta(x) \)

\[ 0 = \frac{\partial F_2}{\partial P} = \frac{\partial F_2}{\partial P}(q, P(q, P)) \]

i.e.

\[ 0 = 0(q, P) \]

together w\( h \) \( \Psi \) above:
\[ P = P(q, P) \]

and these are the equations of our canonical transformation.

We can do the same procedure to find the canonical transformation using two other types of generating functions

\[ F_3 = f_3(q, P) \]

or
\[ F_4 = f_4(P, P). \]

Tabor claims that \( F_2 = f_2(q, P) \) is the most important among the four of them. (Why he claims this? I have nooo idea.)

If the canonical transformation is three-independent, we
If the canonical transformation
\[ \mathbf{Q} = \mathbf{Q}(q, p) \]
\[ \mathbf{P} = \mathbf{P}(q, p) \]
is time-independent, then, the transformed Hamiltonian is:
\[ \tilde{H}(\mathbf{Q}, \mathbf{P}) = H(q(\mathbf{Q}, \mathbf{P}), p(\mathbf{Q}, \mathbf{P})). \]

Now, from the chain rule and the conservation of phase-space volume, we can prove that:
\[ \dot{q} = \frac{\partial H}{\partial p} \]
\[ \dot{p} = -\frac{\partial H}{\partial q} \]

This is to say that, in fact, the transformation \( \tilde{H} \) above is canonical. Indeed, we have:
\[ \dot{q} = \frac{dq}{dt} = \frac{d}{dt} q(p) = \frac{\partial q}{\partial q} \frac{dq}{dt} + \frac{\partial q}{\partial p} \frac{dp}{dt} \]
\[ = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} \]
\[ = \frac{\partial q}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} \]

Noticing that some terms cancel out and rearranging:
\[ = 2 \frac{\partial H}{\partial p} \frac{\partial q}{\partial p} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial q} \]
\[ = \frac{\partial H}{\partial p} \frac{\partial q}{\partial p} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial q} \]
\[ = \frac{\partial H}{\partial p} \frac{\partial q}{\partial p} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial q} \]
\[ \Omega = \frac{\partial f}{\partial p} \frac{\partial (0, p)}{\partial q} \]

and since it is volume-preserving:

\[ \Omega = \frac{\partial f}{\partial p} \]

Similarly,

\[ \dot{p} = \frac{dP(q, p)}{dt} = \frac{\partial P}{\partial q} \dot{q} + \frac{\partial P}{\partial p} \dot{p} = \frac{\partial P}{\partial q} \frac{\partial H}{\partial q} - \frac{\partial P}{\partial p} \frac{\partial H}{\partial p} \frac{\partial H}{\partial q} \left( \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{q}}{\partial p} \right) - \frac{\partial P}{\partial q} \frac{\partial \dot{q}}{\partial p} \frac{\partial \dot{q}}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial \dot{q}}{\partial q} \frac{\partial \dot{q}}{\partial p} \frac{\partial \dot{q}}{\partial p} \]

\[ = -\frac{\partial P}{\partial q} \left( \frac{\partial \dot{q}}{\partial q} - \frac{\partial \dot{q}}{\partial p} \frac{\partial \dot{q}}{\partial p} \right) \]

\[ = -\frac{\partial P}{\partial q} \det \left( \begin{array}{cc} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial p} \\ \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial p} \end{array} \right) \]

Since it is phase-space volume-preserving:

\[ \dot{p} = -\frac{\partial H}{\partial q} \]

Up to this point, we have just considered the case where the transformation is time-independent.

The time-dependent case now follows.
Here then, we consider generating functions of the form:

\[ F_2 = F_2(q, p; t) \]

To find this type of generating function, we will write the action integral:

\[ I(q) = \int_{t_1}^{t_2} L(q, q, \dot{q}) dt \]

in the phase-space:

\[ I[q, p] = \int_{t_1}^{t_2} \left( p \dot{q} - H(q, p, t) \right) dt \]

and require:

\[ \frac{\partial I[q, p]}{\partial \dot{q}} = 0, \quad \frac{\partial I[q, p]}{\partial p} = 0 \]

The variation of \( I[q, p] \) to vanish, while \( q dt = dq \):

\[ I[q, p] = \int_{t_1}^{t_2} p dq - H(q, p, t) dt \]

The same should hold in the phase-space in the new coordinates:

\[ I[\theta, p] = \int_{t_1}^{t_2} p d\theta - H(\theta, p, t) dt \]

We require their total variation to vanish:

\[ S I[\theta, p] = 0 \]
\[ S I[\theta, p] = 0 \]
Hence these two integrals should coincide.

\[ S \left( f_{1}(q, p) - f_{1}(q, p) \right) = 0 \]

Then (3), the difference should be an exact differential:

\[ L[q_{1}(p) - q_{1}(p)] = \int_{t_{1}}^{t_{2}} \frac{df}{ds} \, ds \]

Hence (3):

\[ (pdq - t dt) - (pdq - \dot{q} C(q, p, t) dt) = df \]

where \( f = f(q_{1}, p_{1}, t) \). We can group "differentials"

\[ df = pdq - P \, dq = \left( H(q_{1}, p_{1}, t) - \dot{q} \, C(q, p, t) \right) dt \]

and so:

\[ P = \frac{\partial f}{\partial q} \]

\[ -q = \frac{\partial f}{\partial q} \]

and:

\[ -(H(q_{1}, p_{1}, t) - \dot{q} \, C(q, p, t)) = \frac{\partial f}{\partial t} \]

or:

\[ H(0, p, t) = \frac{\partial f}{\partial t} + H(q_{1}, p_{1}, t) \]

This type of transformation is for the type-I of generating functions, \( f_{1} = f_{1}(q_{1}, 0, t) \).
For generating functions of the type 2i, i.e.:

\[ F_2 = F_2(q,P,t), \]

we re-write (16-x) in the form:

\[ \begin{align*}
    dF &= pdq - Pdq - (H - H') dt \\
    &= pdq - Pdq + d(PQ) - d(PQ) - (H - H') dt \\
    dF + d(PQ) &= pdq - Pdq + d(PQ) - Pdq - (H - H') dt \\
    d(F + PQ) &= pdq + \partial P dq - (H - H') dt
\end{align*} \]

\[ F_2 = F_2(q, P, t) + F + PQ. \]

\[ \therefore \quad dF_2 = pdq + \partial P dq - (H - H') dt \]

Hence:

\[ \begin{align*}
    P &= \frac{\partial F_2}{\partial q} (q, P, t) \\
    \partial P &= \frac{\partial F_2}{\partial P} (q, P, t) \\
    (H - H') &= \frac{\partial F_2}{\partial t} \\
    H(\partial, P, t) &= H(q, P, t) + \frac{\partial F_2}{\partial t}
\end{align*} \]
We go on state with no problem:

\[ H(0, p, t) = H_0(p, t) + \frac{2F_0}{dt} \]

for the four types of generating functions \((s = 1, 2, 3, 4)\).

In the next section, this technique is used to solve some particular Hamiltonian systems.

2.4 Hamilton-Jacobi Equation and Action-Angle Variables.

Goal: Find the canonical transformation that transforms \((q, p) \mapsto (\bar{q}, \bar{p}) = (\bar{q}, p^{(s)})\) where \(p^{(s)}\) is a constant (vector), i.e., the conjugate momenta should be constant as in eqn (2.3.11):

\[ h(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto \bar{H}(p_1, \ldots, p_n) \]

(eqn (34.x) in this notes). \(p_1, \ldots, p_n\) are constants.

We require: Find the suitable generating function.

The suitable generating function will be of the type a function of the old \(q\)'s and a function of new momenta \(p\):

\[ F_2 = F_2(q_1, \ldots, q_n, \bar{p}_1, \ldots, \bar{p}_n) \]

Write (denote): \(S = F_2: \)

\[ S = S(q_1, p^{(s)}) = (q_1, \ldots, q_n, p_1, \ldots, p_n) \]
We denote $P_j = P_j^{(0)}$ with a superscript script, since we wish $P_j = P_j^{(0)}$ to be constants. (Text: $q_j \leftrightarrow P_j^{(0)}$).

Equations (2, 3, 20) (Note: Correct eqs. (41. - x)) Suggest:

(* *)

$$P_j = \frac{\partial S}{\partial q_j}$$  \hspace{1cm} (**) \hspace{1cm} (eqn's (2.4.1.).)

(* *)

$$S_j = \frac{\partial S}{\partial P_j^{(0)}}$$

where $Q_j$ are the new conjugate coordinates corresponding to the $P_j^{(0)}$. Then, the Hamiltonians are related by:

$$H(q_1, ..., q_n, p_1, ..., p_n) = H(P_1^{(0)}, ..., P_n^{(0)})$$

Automatically the right-hand-side is a constant. Thus, using (**), we have:

$$H(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}) = E$$

where $E$ turns to be the total energy of the system.

Solving the Hamilton-Jacobi equation is equivalent to solve the set of characteristic equations, which turn to be the Hamilton equations of motion.
In general, to solve the Hamilton-Jacobi equation turns to be very difficult to solve, unless it is separable.

Notice that:

\[ ds = \sum_{q_1} \frac{\partial S}{\partial q_1} dq_1 + \cdots + \sum_{q_n} \frac{\partial S}{\partial q_n} dq_n = \sum_{i=1}^{n} p_i dq_i + \cdots + p_n dq_n \]

(The quantities, the differentials \( dp_j(0) \) do not appear, since \( p_j(0) \) are constants, although they are variables; that \( S \) depend on: \( S = S(q_1(0), q_2(0), \ldots, q_i(0), p_1(0), \ldots, p_i(0)) \)).

Then, \( S \) is the line integral:

\[ S = \int_{\mathbf{q}_0}^{\mathbf{q}_1} p_1 dq_1 + \cdots + p_n dq_n. \]  \( (\mathbf{x}) \)

with

\[ \mathbf{q}_0 = (q_1(0), q_2(0), \ldots, q_n(0)) \] - initial position

\[ \mathbf{q}_1 = (q_1(t), q_2(t), \ldots, q_n(t)) \] - position of the particle-system at \( t = t \).

This formula is not so useful: To use \( S \), we require to know \( q(t) \), the solution of the problem should be known a priori.