2.4.2. The Hamilton-Jacobi equation for one-degree of freedom systems.

For one-degree of freedom systems, the Hamilton-Jacobi is easy to solve.

The Hamilton-Jacobi equation is

$$ H(q, p) = H(p^{(0)}) $$

which is just the substitution $p^{(0)} = p^{(0)}(q, p)$, but $H$ only depends on new variables by the definition of $p_i$, i.e., $p = \frac{\partial s}{\partial q}$; we have the P.D.E.

$$ H(q, \frac{\partial s}{\partial q}) = \tilde{H}(p^{(0)}) $$

The RHS is just a constant, and we set it as $\alpha$,

$$ H(q, \frac{\partial s}{\partial q}) = \alpha $$

or

$$ H(q, \frac{\partial s}{\partial q}) = E $$

the total energy of the system. The relations for the generating functions are:

$$ p = \frac{\partial s(q, p^{(0)})}{\partial q} \quad \text{and} \quad 0 = \frac{\partial s(q, p^{(0)})}{\partial p^{(0)}} $$

$$ = S(q, p^{(0)}). $$
Since we require a canonical transformation:

\[ \mathbf{p}^{(0)} = \mathbf{E} = -\frac{\partial H}{\partial \mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \]

We know:

\[ H(p^{(0)}) = E. \]

(\(\phi\)) Take \( p_{(0)} = E \) hence \( H(E) = E \)

and \( \mathbf{0} = \frac{\partial H(E)}{\partial E} = \frac{\partial E}{\partial E} \]

Hence:

\[ p^{(0)} = \text{constant} = E \] for the total energy of the system.

Notice this:\n
(\(\phi\)) \( (\mathbf{x}, x) \)

\[ t - t_0 = \mathbf{0} = \frac{\partial S}{\partial p^{(0)}} = \frac{\partial S}{\partial (p, E)} \]

(\(\mathbf{x}, x\))

\( \partial E \int_0^1 p(q', E) dq' = \int_0^1 \frac{\partial p(q', E)}{\partial E} dq' \]

(\(\mathbf{x}, x\))

\[ = \int_0^1 \frac{\partial p(q', E)}{\partial E} dq' \] Assuming \( S \in C^2(\mathbb{R}^2) \)
For a portfolio in a simple potential, we have the Hamiltonian
\[ H(q, p) = \frac{1}{2m} p^2 + V(q). \]

Here: \( H(q, p) = \hat{H}(E) = E \)

so that:
\[ p(q, E) = \pm \sqrt{2m(E - V(q))}, \]

and thus, by (32.1**):
\[ t - t_0 = \pm \int_{q_0}^{q} \frac{1}{\sqrt{2m(E - V(q))}} \, dq \]

\[ = \pm \sqrt{\frac{m}{2}} \int_{E}^{E - V(q)} \, dq. \]

This exercise is particularly useful to find the particularly useful action-angle variables, so they are extremely useful to describe many degrees of freedom systems.
2.4.6 **Action-Angle Variables for One Degree of Freedom**

Let us consider "bounded Hamiltonians." Then, the trajectories in phase-space are closed, periodic trajectories.

This means that, after a period of time, if we start at a point \((q, p)\), we return to the same point:
\[
T = \frac{2\pi}{\omega}, \quad \text{w-frequency of motion,}
\]

The idea is that the conjugate variable (to the constant new momentum) changes by \(2\pi\) after a period of motion.

The new momentum is denoted by: \(I\)

The new conjugate variable is denoted by: \(\Theta\), and they are \((I, \Theta)\), the action-angle variables.

We want \(I\) to be constant.

We proceed as follows: The generating function relations are
\[
p = \frac{\partial S}{\partial q}\]
\[
\Theta = \frac{\partial S}{\partial I}
\]

Comparing \((4.3x)\), \((47.4x)\) or \((51.4x)\)
The Hamilton-Jacobi equation is given by:

\[ H(q, \frac{\partial S}{\partial q}) = E. \]

For a path \( \tilde{H}(t) = E \), we have (from \( S+\cdot dS \)):

\[
\frac{d}{dq} \theta = \frac{d}{dq} \left( \frac{2S}{I} (q, I) \right) = \frac{2}{\delta I} \left( \frac{2S}{I} \right)
\]

Let us denote by \( \mathcal{C} \), the curve \( H(q, p) = E \)

i.e. \( \tilde{H}(I) = E \). We are requiring to change when we complete a turn around \( C \). Then:

\[
2\pi = \oint_C d\theta = \oint_C \frac{2}{\delta I} \left( \frac{2S}{I} \right) dq = \frac{2}{\delta I} \oint_C \frac{2S}{I} dq.
\]

i.e.

\[
2\pi = \frac{2}{\delta I} \oint_C P \, dq \quad (\ast)
\]

If we define:

\[
\oint_C P \, dq \equiv 2\pi I.
\]

then \( (\ast) \) holds. Actually the definition of the action variable is given by:

\[
I = \frac{1}{2\pi} \oint_C P \, dq.
\]
The Hamilton-Jacobi equation is solved by:

\[ S = \int_{q_0}^{q} p(q', E) dq' \]

which follows from \((41, x)\), where \(S = S(q, E)\). Since \(p = \frac{\partial S}{\partial q}\), then, the action integral is defined by:

\[ I = \frac{1}{2\pi i} \oint p(q, x) dq \]

around \( C = \left\{ (q, p) \in \text{Phase-space} \mid H(q, p) = E = \text{const} \right\} \). Thus, the new Hamiltonian is computed as:

\[ \tilde{H}(I) = H(q(I, 0), p(I, 0)) \]

and

\[ \dot{\theta} = \frac{2H}{\dot{I}} \]

\[ \dot{I} = -\frac{2H}{\theta} \]

i.e.,

\[ \dot{\theta} = \frac{2H}{\dot{I}} = \frac{\tilde{H}}{I} \]

\[ \dot{I} = 0 \Rightarrow I = I_0 \]

By def'n:

\[ \omega(I) = \frac{2H}{\dot{I}} \]

= 56
Hence: \[ I = I_0 \] is a constant, so well as \( \omega(t) = \omega(T_0) \) is a constant. Hence:

\[ \dot{\theta} = \frac{2\pi}{T} (T_0) = \omega(T_0) \]

\[ \Rightarrow \theta = \omega(T_0) t + \Theta. \]

Example: The harmonic oscillator.

\[ H(q, p) = \frac{1}{2m} p^2 + \frac{1}{2} k q^2 \quad \ldots (I) \]

The Hamilton-Jacobi equation is given by:

\[ \frac{1}{2m} \left( \frac{\partial \Sigma}{\partial q} \right)^2 + \frac{1}{2} k q^2 = E \]

(w is such that: \( \dot{q} = -\omega^2 q \), or \( \dot{q} = -\frac{k}{m} q \))

i.e., \( \omega^2 = \frac{k}{m} \). In Tabor's text: \( m = 1 \).

From (I):

\[ I = \frac{1}{2\pi} \int_C \sqrt{2m(E - \frac{1}{2} k q^2)} \, dq \]

is the action variable. \( C \) is the closed path.

\[ H(q, p) = E \] between the turning points (where \( p = 0 \))

\[ q_{1/2} = \pm \sqrt{\frac{2E}{k}} = \pm \sqrt{\frac{2E}{k} \frac{1}{\omega}} \]

\( \approx 57.1 \)
Integrating \((57 \times x)\)

\[
I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{2m}}{\sqrt{E - \frac{1}{2} k q^2}} dq.
\]

\[
= \frac{4\sqrt{2m}}{2\pi} \int_{0}^{\pi} \sqrt{E - \frac{1}{2} k q^2} dq
\]

Let \(\frac{1}{2} k q^2 = E \cos^2 \theta\)

\[
i.e. q = \sqrt{\frac{2E}{k}} \cos \theta
\]

\[
(\Rightarrow q(0) = \sqrt{\frac{2E}{k}}, \quad q(\pi) = -\sqrt{\frac{2E}{k}})
\]

\[
= \frac{2}{\pi} \sqrt{2m} \frac{\sqrt{2E}}{\sqrt{k}} \int_{0}^{\pi} \sqrt{E - E \cos^2 \theta} \sqrt{\frac{2E}{k}} (\sin \theta) d\theta
\]

\[
= \frac{2}{\pi} \sqrt{2m} \sqrt{\frac{2E}{k}} \int_{0}^{\pi} \sqrt{1 - \cos^2 \theta} \sin \theta d\theta
\]

\[
= \frac{2 \cdot 2}{\pi} \sqrt{mE} \int_{0}^{\pi} (\sin \theta) \sin \theta d\theta
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} \sin^2 \theta d\theta
\]

\[
= \frac{\pi}{2} \int_{0}^{\pi} \sin^2 \theta d\theta
\]

\[
= \frac{\pi}{2} \int_{0}^{\pi} \frac{1}{2} (1 - \cos 2\theta) d\theta
\]

\[
= \frac{\pi}{2} \left[ \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_{0}^{\pi}
\]

\[
= \frac{\pi}{2} \left( \frac{\pi}{2} - 0 \right)
\]

\[
= \frac{\pi^2}{4}
\]

\[
= 58.\overline{9}
\]
\[ I = \frac{4}{\pi \omega} \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} \, d\theta \]

\[ = \frac{4E}{\pi \omega} \left( \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \bigg|_0^{\pi/2} \]

\[ = \frac{4E}{\pi \omega} \left( \frac{\pi}{4} - \frac{\sin \pi}{4} \right) - (0 - 0) \]

\[ = \frac{4E}{\pi \omega} \frac{\pi}{4} \]

\[ = \frac{E}{\omega} \]

\[ \Rightarrow \quad I = \frac{E}{\omega} \]

is the new momentum, the action variable.

The new Hamiltonian is given by:

\[ \tilde{H}(I) = H(q, p) = E \]

\[ = E = \text{total energy} \]

\[ = \omega \, I \]

\[ \Rightarrow \quad \tilde{H}(I) = \omega \, I \]

is the new Hamiltonian.
This is
\[
\begin{align*}
0 &= \omega \\
I &= 0
\end{align*}
\]

Rule: since \( \frac{\partial H}{\partial I} = \omega(I) \) by definition,
\[\omega(I) = \omega\]

coincides with the frequency of the harmonic oscillator. End of Rule.

We now write the action (the generating functional) as a function of \( q \) and \( I \) is:
\[
S(q, I) = \int_q^q P(q', I') \, dq'
\]
\[
= \int_{q_0}^q \sqrt{2m \left( E - \frac{1}{2} k q'^2 \right)} \, dq'.
\]
\[
= \int_{q_0}^q \sqrt{2m \left( \omega I - \frac{1}{2} k q'^2 \right)} \, dq'.
\]
\[
k = mw^2
\]
\[
= \int_{q_0}^q \sqrt{2m \left( \omega I - \frac{1}{2} mw^2 q'^2 \right)} \, dq'.
\]
\[
= \sqrt{2m} \omega \int_{q_0}^q \sqrt{I - \frac{1}{2} mw q'^2} \, dq'.
\]
Again: \[ \frac{1}{2} \mu w q^2 = I \cos^2 \phi' \]
\[ q' = \sqrt{\frac{2I}{\mu w}} \cos \phi' \]

Hence:
\[
S(q, I) = \sqrt{2\mu w} \int_q^{q_0} \sqrt{I} \sqrt{1 - \cos^2 \phi'} \sqrt{\frac{2I}{\mu w}} \, d\phi'
\]
\[
= \sqrt{2\mu w} J \sqrt{\frac{2I}{\mu w}} \int_{\phi_0}^{\phi} \sin \phi' \sin \phi' \, d\phi'
\]

Assume \( 0 < \phi_0 < \phi < \phi' \) \( \phi \) \( \phi' \) \( \pi/2 \) \( \Rightarrow \) \( |\sin \phi'\sin \phi'\phi| = |\sin \phi'\phi| \)
\[
= 2I \int_{\phi_0}^{\phi} \sin^2 \phi' \, d\phi'
\]
\[
= 2I \int_{\phi_0}^{\phi} \frac{1 - \cos 2\phi'}{2} \, d\phi'
\]
\[
= 2I \left[ \left. \left( \frac{\phi'}{2} - \frac{\sin(2\phi')}{4} \right) \right|^{\phi}_{\phi_0} \right]
\]
\[
= I \left( 2\phi' - \sin(2\phi') \right) \bigg|_{\phi_0}^{\phi}
\]
\[
= I \left( 2(\phi - \phi_0) - (\sin(2\phi - 2\phi_0)) \right)
\]

... which is not so nice to write in terms of \( q' \):

\[ q' = \sqrt{\frac{2I}{\mu w}} \cos \phi' \]
We can better use: \( f(t, x) \)

\[
\Theta = \frac{\partial}{\partial t} S(t, 1) = \frac{\partial}{\partial t} \int_{q_0}^{q} \sqrt{2m \omega} \sqrt{I - \frac{1}{2} m \omega q'^2} \, dq'
\]

\[
= \sqrt{2m \omega} \int_{q_0}^{q} \frac{1}{2} \sqrt{I - \frac{1}{2} m \omega q'^2} \, dq'
\]

\[
= \sqrt{2m \omega} \int_{q_0}^{q} \frac{d}{2} \sqrt{1 - \frac{1}{2} m \omega q'^2} \, dq'
\]

Proposition:

\[
\frac{1}{2} m \omega q'^2 = I \cos^2 \phi
\]

\[
q' = \sqrt{\frac{2I}{m \omega}} \cos \phi
\]

\[
\frac{dq'}{d\phi} = \sqrt{\frac{2I}{m \omega}} (-\sin \phi)
\]

\[
\Theta = \sqrt{2m \omega} \int_{\theta_0}^{\phi} \frac{1}{2} \sqrt{I - \cos^2 \phi} \sqrt{\frac{2I}{m \omega}} (-\sin \phi) \, d\phi
\]

If \( \sin \phi > 0 \):

\[
\Theta = \phi - \theta_0 \Rightarrow \phi = \Theta + \theta_0
\]

\[
\cos \phi = \cos(\Theta + \theta_0) \Rightarrow q = \sqrt{\frac{2I}{m \omega}} \cos(\Theta + \theta_0)
\]

\[
= 62 =
\]
Now, we know:

\[ \theta = \frac{\dot{\theta}}{\dot{\theta}} = \omega(I) = \omega : \]

and \( \omega = \sqrt{\frac{k}{m}} \) is a constant.

\[ \theta = \omega t : \]

so we have:

\[ q = \sqrt{\frac{2I}{m\omega}} \cos (\omega t + \theta_0) : \]

and this is the solution to harmonic oscillators.

If we consider the Hamiltonian:

\[ H = \frac{1}{2m} p^2 + \frac{1}{4} \beta q^4 : \]

we can compute the action, \( I \), and the Hamiltonian in \( n \) free-angle-variables, \( H(I) \).

We have:

\[ I = \frac{\sqrt{2}}{3 \sqrt{\pi^3}} \beta^{1/4} \left( \frac{7^{9/8} \Gamma(\frac{1}{4})}{4} \right) E^{rac{34}{4}}. \]

Roughly, \( W(I) = \frac{2H}{\dot{I}} \) is

\[ \text{new} \ L \text{-dependent} \]

\[ \Rightarrow \text{We have a NL problem} \]

\[ f(I) = I^{4/3} \left( \frac{3^4 \pi^6 \beta}{2^2 \Gamma^8(\frac{1}{4})} \right) \]

We can do the exercise here, but let us keep going and
do the exercise for the system of our interest, i.e.,
the 2D hex lattice.
20.5 INTEGRABLE HAMILTONIANS.

Systems of one degree of freedom can be solved by using the techniques of ODE's or by the Hamilton-Jacobi equation. But, what if the systems is of two or more degrees of freedom?

25.2 SEPARABLE SYSTEMS.

The Hamilton-Jacobi equation (the stationary case):

\[ H(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}) = \hat{H}(x_1, \ldots, x_n) \]

is, in general, not one able of being solvable.

One of the cases in which it can be solvable is when the eqn is separable; i.e., if the generating function can be written as:

\[ S(q_1, \ldots, q_n, x_1, \ldots, x_n) = \sum_{j=1}^{n} S_j(q_j, x_1, \ldots, x_n) (\star) \]

A particularly case of this is when we can write the Hamiltonian as:

\[ H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{j=1}^{n} H_j(q_j, p_j) (\star \star) \]

(e.g. uncoupled oscillators);

\[ =64= \]
Then, the Hamilton-Jacobi reduces to:

$$H(q_k, \dot{q}_k) = \alpha_k, \quad k=1,2,\ldots,n \quad (*)$$

Here,

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = \alpha,$$

with

$$\alpha = \tilde{H}(I_1, I_2, \ldots, I_n).$$

and $\tilde{H}$ is the transformed Hamiltonian. (Strictly, we will say how to write $H$ in terms of $I_k$'s.)

For example of separable generating function, $S(q, p)$, see London-Hitschutz.

From the definition of the momenta,

$$P_k = \frac{\partial S}{\partial q_k},$$

and with the help of separable generating function, i.e., eqn. (6A.4):

$$P_k = 2 \sum_{j=1}^{n} S_j(q_j; \alpha_j, \ldots, \alpha_n).$$

i.e.,

$$P_k = \frac{2}{\partial q_k} S_j(q_k; k_1, \ldots, \alpha_n). \quad (**)$$

From this expression for $P_k$ we can compute the action variables

$$I_k = \frac{1}{2\pi} \oint P_k(q_k; k_1, \ldots, \alpha_n). \quad (***)$$

$$\text{action variables}$$

$$I_k = \frac{1}{2\pi} \oint P_k(q_k; k_1, \ldots, \alpha_n).$$

$$= 65 =$$
where the \( \ell_k \)'s are closed paths for the periodic motion in the system.

Now, from (65.0.0.0) we can solve the \( \omega_k \)'s in terms of the \( I_k \):

\[
\alpha_k = \omega_k \left( q_k, I_1, I_2, \ldots, I_n \right) \quad (k = 1, 2, \ldots, 3)
\]

(provided the Inverse Function Theorem works).

Since:

\[
S_0 = S_0 \left( q_0, x_1, \ldots, x_2 \right)
\]

and if we substitute (x) into \((**)*\), we have:

\[
\tilde{S} = \sum_{j=1}^{m} S_0 \left( q_j, x_1 \left( q_1, I_1, \ldots, I_n \right), \ldots, \omega_1 \left( q_n, I_1, \ldots, I_n \right) \right)
\]

10.

\[
\tilde{S} = S \left( q_1, q_2, \ldots, q_n, I_1, I_2, \ldots, I_n \right)
\]

and the angle variable is:

\[
\Theta_k = \frac{\partial \tilde{S}}{\partial I_k} = \sum_{j=1}^{n} \frac{\partial \tilde{S}_j}{\partial I_k} \left( q_j, q_1, \ldots, q_n, I_1, \ldots, I_n \right)
\]

Then, solving \( q_k \) from (***), in terms of \( \Theta_k, I_k \), we have:

\[
q_k = q_k \left( \Theta_1, \ldots, \Theta_n, I_1, \ldots, I_n \right)
\]

Substituting these \( q_k \)'s (eq's (**)) into (65.0.0.0)

\[
= 66
\]
we have:

\[ \mathbf{R}_k = \mathbf{R}_k (\mathbf{Q}_1, \ldots, \mathbf{Q}_n, I_1, \ldots, I_n) \quad (\text{a}) \]

Hence:

\[ \mathbf{H} (I_1, \ldots, I_n) = \mathbf{H} (\mathbf{Q} (\mathbf{Q}_1, \ldots, \mathbf{Q}_n, I_1, \ldots, I_n), \mathbf{P} (\mathbf{Q}_1, \ldots, \mathbf{Q}_n, I_1, \ldots, I_n)) \]

\[ \text{is the transformed Hamiltonian, and the Hamilton's qu's are:} \]

\[ I_k = \frac{\partial \mathbf{H} (I_1, \ldots, I_n)}{\partial \mathbf{Q}_k} = \mathbf{w}_k (I_1, \ldots, I_n) \]

\[ \mathbf{Q}_k = -\frac{\partial \mathbf{H} (I_1, \ldots, I_n)}{\partial I_k} \]

Since \( \frac{\partial \mathbf{H}}{\partial \mathbf{Q}_k} = 0 \), \( \Rightarrow I_k \) are constants \( \Rightarrow \mathbf{H} (I_1, \ldots, I_n) \)

\[ \text{are also constants; } \Rightarrow \mathbf{w}_k (I_1, \ldots, I_n) \text{ are also constants} \]

\[ \Rightarrow \]

\[ I_k = \text{const}; \]

\[ \mathbf{Q}_k = \mathbf{w}_k (I_1, \ldots, I_n) t + \mathbf{Q}_k (0) \]

where \( \mathbf{Q}_k (0) \) are constants of integration (are known to us), and they are arbitrary.

The constants \( \mathbf{Q}_k (0) \) together with the \( I_k \)'s (n integrals of motion) constitute a set of 2n constants, enough to integrate the problem.
Tabor claims: "The Jv's are rather special set of constants. Once we have them, the symplectic structure of the Hamiltonian immediately gives us the other set."

What does he mean by the "symplectic structure of the Hamiltonian system immediately "gives" (3) us the other set."
SUMMARY FOR THE Hamilton-Jacobi Equations

We wish to canonical transformation

\[(q, p) \rightarrow (\bar{q}, \bar{p}) = (0, \bar{p}^0)\]

such that \(P = \bar{p}^0\) is a constant of motion.

Since the transformation is canonical:

\[\dot{q} = \frac{\partial H}{\partial \dot{p}} \quad \text{(A)}\]

\[\dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \bar{H} = \bar{H}(\bar{p}) \text{ only} \quad \text{(**)i)\}

But we want \(\bar{P} = \bar{p}^0\) is constant \(\Rightarrow \bar{P} = 0\).

\[\Rightarrow \quad 0 = \frac{\partial \bar{H}}{\partial \bar{q}} \quad \text{and} \quad \bar{H} = \bar{H}(\bar{p}) \text{ also a constant and, equa(A)}\]

can be easily integrated. Hence:

\[H(q, p) = \bar{H}(\bar{p}^0) \quad \text{(***)}\]

is a constant of motion.

Now, we look for generating function of the type - 2, i.e.

\[F_2 = S(q, p)\]

By relations (**)ii)

\[p = \frac{\partial S}{\partial q}(q, p) \quad \text{(**)}\]

we then know:

\[0 = \frac{\partial S}{\partial \dot{q}}(q, p) \quad \text{(**ii)}\]

\[= 6q = \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \}
Using (69.***), into (69.*), we get:

\[ h(q, \frac{\partial S}{\partial q}) = E \]  

where \( E = H(P, q) \) is the total energy. Equation (*) above defines a P.D.E. for the generating function \( F_2 S(q, P) \). Assume we can solve (*) (within the "appropriate" boundary conditions). So, we get the solution \( F_2 S(q, P) \).

By equations (69.***), and (69.****), we obtain

\[ p = \frac{\partial S}{\partial q} \]

\[ \Omega = \frac{\partial S}{\partial P} \]

\( \Omega \) and \( p \) are functions of \( q \) and \( P \):

\[ p = f_2 (q, P) \quad \text{--- (***1)} \]

\[ \Omega = f_1 (q, P) \quad \text{--- (***2)} \]

If \( \frac{\partial f_2}{\partial P} \neq 0 \) \( \Rightarrow \)

\[ P = f_2^{-1} (q, P) \quad \text{--- (***3)} \]

Substitute (**3) into (**2) to get

\[ \Omega = f_1 (q, f_2^{-1} (q, P)) \quad \text{--- (**4)} \]

\[ q = f_1 (q, f_2^{-1} (q, P)) \quad \text{--- (**5)} \]

and these functions define a canonical transformation.
In the usual notation of action-angle (P=Q) variables we have:

Angle variables: \( \theta = \Theta = \tilde{f}_1(q,p) \)

Action variables: \( I = P_\theta = \tilde{f}_2(q,p) \)

Notice that eqn (4) is a constant of motion. If we are in a \( \mathbb{R}^{2n} \)-phase space: \( (q_1, \ldots, q_n, p_1, \ldots, p_n) \).

Angle variables:
\[ \theta^j = \Theta^j = \tilde{f}_1(q^j, p) \]

Action variables:
\[ I^j = P_{\theta^j} = \tilde{f}_2(q^j, p) \]

Equation (***) constitutes the system of \( n \) integrals of motion to completely integrate the problem.

What about \( \tilde{H} = \tilde{H}(I^j) \)? Is this an unnecessary integral of motion? Or is \( \tilde{H} = \tilde{H}(I^j) \) one of the \( I^j \)'s?
2.5.6 Properties of Integrable Systems.

Given a Hamiltonian system, we do not know if it is integrable or not. If this Hamiltonian system is in $\mathbb{R}^{2n}$, we know that we require $2n$ integrals of motion, which are all independent (i.e., their gradients should be linearly independent).

If these are the $n$ conjugate momenta, $I_k$, Hamiltonian's equations are trivially integrated, although the transformation back to $q(p)$ is not trivial in general.

Moreover, we can integrate the problem if one finds $n$ independent paths $C_k$, $k = 1, 2, \ldots, n$ to define the action variables:

$$I_k = \frac{1}{2\pi} \oint_{C_k} P_k(q) dq,$$

($dq = dq_1 dq_2 \ldots dq_n$), and, in this case, it is not necessary to have a separable generating function $F_2 = S(q, p)$. 

$= 73$
If $F_0(1; \theta)$ is an integral of motion, then

$$F_0(q(1), p(1)) = f_J$$

with $f_J$ is a constant (independent of $\theta$).

In Poisson bracket notation:

$$[F_1, F_2] = 0.$$ 

A Hamiltonian system is said to be completely integrable if there exists $n$ integrals of motion,

$$F_1, F_2, \ldots, F_n.$$ 

Say $F_1 = \tilde{H}(I_1)$, which are all in involution:

$$[F_{i}, F_{j}] = 0 \quad \text{for} \quad i, j = 1, 2, \ldots, n.$$ 

Additionally, these integrals should be independent, in the sense that all their gradients $\nabla F_j$ are linearly independent, with $\nabla F_j = \left( \frac{\partial F_j}{\partial q_1}, \ldots, \frac{\partial F_j}{\partial q_n}, \frac{\partial F_j}{\partial p_1}, \ldots, \frac{\partial F_j}{\partial p_n} \right)$.

The existence of $n$ integrals of motion $F_k$, $k = 1, 2, \ldots, n$, means that, except for the trajectories $q_j(\theta), p_j(\theta)$ ($j = 1, 2, \ldots, n$), move in the $2n$ phase-space, they are restricted to a $2n - n = n$ dimensional manifold $M$. 

$= 74=$
Proposition: The manifold \( \mathcal{M} \) has the topology of an \( n \)-dimensional torus.

Proof: Define the following vector fields,
\[
\dot{\mathbf{e}}_k = J \frac{\partial}{\partial \mathbf{F}_k}, \quad k = 1, 2, \ldots, n,
\]
where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) and \( I \) is the \( nxn \) identity matrix, and \( \frac{\partial}{\partial \mathbf{F}_k} \) is defined as before (page 74).

If \( F_1 = H \) is the Hamiltonian, then:
\[
\dot{\mathbf{e}}_1 = \begin{pmatrix} \frac{d}{dt} \\ \frac{d}{dt} \end{pmatrix}
\]
by the Hamilton equations. Then, \( \dot{\mathbf{e}}_1 \) is tangent to the manifold \( \mathcal{M} \). Hence, it just plays the role of a parameter. Similarly, define the parameters \( \dot{\mathbf{e}}_k \), \( k = 2, 3, \ldots, n \), with \( \dot{\mathbf{e}}_1 = \dot{t} \). Then, define:
\[
\frac{d \dot{\mathbf{e}}}{dt} = \frac{d}{dt} \frac{\partial}{\partial \mathbf{F}}.
\]
All of these must be tangent to \( \mathcal{M} \), and the trajectories \( \mathbf{q}(t, \mathbf{e}) \) and \( \mathbf{p}(t, \mathbf{e}) \) lie on the manifold \( \mathcal{M} \). To see this, we follow the argument.
The gradient $\nabla q F_e$ must be orthogonal to the manifold $M$. Then, $J \nabla q F_e$ should be tangent to the manifold, since $J \nabla q F_e$ is orthogonal to $\nabla q F_e$. This is true because, for $n = 3$:

$$(\nabla q F_e, J \nabla q F_e) = \begin{pmatrix} \frac{\partial F}{\partial q} \\ \frac{\partial F}{\partial \theta} \\ \frac{\partial F}{\partial \phi} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial F}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial q} \\ \frac{\partial F}{\partial \theta} \\ \frac{\partial F}{\partial \phi} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{\partial F}{\partial \phi} \end{pmatrix} = 0;$$

Then, $J \nabla q F_e$ is tangent to $M$.

From this follows that, since $\dim(M) = n$, there must be at most $n$ linearly independent tangent vectors, i.e., if $1 \leq k \leq K_0$, then $J \nabla q F_e \quad 1 \leq k \leq K_0 \leq n$.

Then, there must be $n$ integrals of motion.

Thus, all the vector fields $\mathbf{E}_e = J \nabla q F_e, \quad e = 1, 2, \ldots, n$ are tangent to $M$.

The fact that $\nabla q F_e$ are linearly independent implies that the $\mathbf{E}_e$'s are linearly independent.
Since all the vector fields on the manifold M, this manifold should be equivalent to an n-dimensional torus, by the Hopf-Poincaré theorem. Q.E.D.

We should review what this theorem says.

Question Now, I am wondering about that systems which have unbounded trajectories. They are not periodic, as this theory is requiring.

The existence of this torus in phase-space provides the means to define the action variables in a "representation-independent" way. (Textually taken from Tabor's book.) What does Tabor mean by this ???

The n-torus is a periodic object and can be considered as the direct product of n-independent 1-torus (S^1, the 1-sphere). (Tabor calls them 2π-periodicities.)
This means that we can find $n$ different, independent paths, $\gamma_k, k=1,2,..., n$ on the torus (which cannot be deformed into each other, neither can be deformed into a point).

We then define the action

$$I_k = \frac{1}{2\pi} \oint \sum_{\gamma_k} \frac{1}{m_k} \text{P}_n \, dq_n.$$

Now, from the generating function $S(q_1, ..., q_n, I_1, ..., I_n)$, one can obtain the conjugate-angle variables

$$\psi_k = \frac{2}{\pi} \frac{\partial S}{\partial I_k} (q_1, ..., q_n, I_1, ..., I_n)$$

(Tabor uses $\theta_k$; Arnold, $\psi_k$).

Then, the Hamilton's equations in the action-angle variables are:

$$\frac{d}{dt} I_k = -\frac{\partial H}{\partial \psi_k}$$

$$\frac{d}{dt} \psi_k = \frac{\partial H}{\partial I_k} = \frac{d}{dt} \left( I_1, ..., I_n \right)$$

Remark: If the system is integrable, the transformation to action-angle variables is "global" (?)
"This is to say, the phase space is filled with tori (although there could be some separatrices)."

"Therefore, the trajectory will stay in a torus, or in another."

* Given initial condition:

\[ (q_1(0), q_2(0), \ldots, q_n(0), p_1(0), p_2(0), \ldots, p_n(0)) \]

determines the integrals of motion

\[ f_k = F_k(q(0), p(0)) = F_k(q(t), p(t)) \]

(?) This set of constants of motion \( f_k = F_k(q(t), p(t)) \)

determines the torus on which the motion is performed.

(i.e., they determine the values of the action variables \( I_k, k = 1, \ldots \))

* The value of \( \Theta_k \in (0, \pi) \) determines the position of the particle on that torus.

Case when the Hamiltonian is conserved:

\[ H(q, p) = E, \quad \text{with} \quad q \in \mathbb{R}^n, \quad p \in \mathbb{R}^n \]

and that it is completely integrable, then:

- Phase-space: dim 2n-dimensional
- Energy manifold: \((2n-D)\)-dimensional
- Torus: \(n\)-dimensional
From this, we can deduce the following:

1) For 1-degree of freedom systems \((n = 1)\), the torus coincides with the energy manifold. Both of them are 1-dimensional.

   This means that the system is "ergodic" i.e., any trajectory "uniformly" explores (?) (covers ?) the 1-dim energy shell.

2) For \(n = 2\): the 2-dim torus is embedded in the 3-dim Energy manifold. Then, the torus divides the energy manifold in inside and outside.

   (?) Then, if there is a "gap" (?) between two tori (defined by two different energies (?) ), then the trajectory cannot escape from that "gap".

3) For \(n \geq 3\), trajectories in "gaps" (?) between higher-dimensional tori can escape (move (?) ) to other regions of the energy manifold.

   This is called "Arnold diffusion". (Does this have to deal with the FPU phenomenon ?)
2.5.3 Examples of Integrable Systems

The two-dimensional harmonic oscillator:

\[ H(q_1, p_1) = \frac{1}{2} (p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2) \]

The two integrals of motions are:

\[ I_1(p_1, q_1) = \frac{1}{2} (p_1^2 + \omega_1^2 q_1^2) = E_1 \quad \text{constant} \]

\[ I_2(p_2, q_2) = \frac{1}{2} (p_2^2 + \omega_2^2 q_2^2) = E_2 \]

This system is separable, therefore, integrable.

The motion consists of independent librations in the \((p_1, q_1)\) and \((p_2, q_2)\) planes, and these actions can be computed as follows:

\[ I_1 = \frac{1}{2\pi} \oint_{C_1} p_1 \, dq_1(q_1; E_1, E_2) \]

\[ I_2 = \frac{1}{2\pi} \oint_{C_2} p_2 \, dq_2(q_2; E_1, E_2) \]

where the circuits \(C_1\) and \(C_2\) are round.

Trips between the librational turning points \(\pm \sqrt{2E_1^1/\omega_1}\) and \(\pm \sqrt{2E_1^1/\omega_1}\) in the \((q_1, p_1)\) plane,

and \(\pm \sqrt{2E_2^1/\omega_2}\) and \(\pm \sqrt{2E_2^1/\omega_2}\) in the \((q_2, p_2)\) plane.

\[ = 84 = \]
The Hamiltonian in the action-angle variables is

\[ H(I_1, I_2) = w_1 I_1 + w_2 I_2. \]

For conservative (two-degree-of-freedom) systems, let us draw trajectories of constant energy \( E = H(I_1, I_2) \) in the \( I_1, I_2 \)-plane.

These are easy to draw. The equation

\[ w_1 I_1 + w_2 I_2 = E \]

represents a straight line with slope \(-\frac{w_2}{w_1}\) and

y-intercept \( \frac{E}{w_1} \) (\( w_1 \neq 0 \)).

Since \( I_1, I_2 \) are constant of motion, the point \((I_1, I_2)\) on the line with energy \( E \) (i.e., y-intercept \( \frac{E}{w_1} \)) represents a torus (a particular torus) in the phase-space.
Example 2. Particle constrained on a finite box in the \((x,y)\)-plane.

Here, we consider a particle of mass \(m\), confined to a box in the \((x,y)\)-plane with Hamiltonian:

\[
H(q, p) = \frac{1}{2m} (p_1^2 + p_2^2).
\]

Since \(2H = 0\) (i.e., the coordinates \(q_1, q_2\) do not appear in the Hamiltonian) the momenta \(p_1\) and \(p_2\) are conserved quantities, therefore the Hamiltonian is conserved.

Consider that the particle is in the box:

\[
\begin{align*}
0 &< x < a, \\
0 &< y < b.
\end{align*}
\]

Then, the integrals can be easily computed.

\[
I_1 = \frac{1}{2\pi} \int_{C_1} p_1 \, dq_1 = \frac{1}{2\pi} \int_{C_1} p_1 \, dq_1 = \frac{1/2a}{2\pi} = \frac{1/2a}{2\pi} \Rightarrow I_1 = \frac{a p_1}{2\pi}
\]

\[
I_2 = \frac{1}{2\pi} \int_{C_2} p_2 \, dq_2 = \frac{1}{2\pi} \int_{C_2} p_2 \, dq_2 = \frac{1/2b}{2\pi} = \frac{1/2b}{2\pi} \Rightarrow I_2 = \frac{b p_2}{2\pi}.
\]
Hence, the Hamiltonian is transformed to:

\[ H(q, p) = \frac{1}{2} m \left( \frac{q^2}{a^2} + \frac{p^2}{b^2} \right) = \frac{1}{2 \mu} \left( \left( \frac{\pi I_1}{a} \right)^2 + \left( \frac{\pi I_2}{b} \right)^2 \right) \]

\[ = \frac{\pi^2}{2 \mu} \left( \frac{I_1^2}{a^2} + \frac{I_2^2}{b^2} \right) = \tilde{H}(I_1, I_2) \]

Hamilton's equations are:

\[ \dot{I}_1 = \omega_1 = \frac{\partial H}{\partial \dot{I}_1} = \frac{\pi^2}{\mu a^2} I_1 \]

\[ \dot{I}_2 = \omega_2 = \frac{\partial H}{\partial \dot{I}_2} = \frac{\pi^2}{\mu b^2} I_2 \]

We notice a dependence on the actions, meaning that the system is **non-linear** (or the boundary opposes the nonlinearity).

\[ \vec{\nabla} \tilde{H} = \nabla \tilde{H} = E - \text{const.} \]

Note that \( \vec{\nabla} \tilde{H} \) changes along \( \tilde{H}(I_1, I_2) = E \).

Since \( \vec{\nabla} \tilde{H} = \left( \begin{array}{c} \frac{\partial \tilde{H}}{\partial I_1} \\ \frac{\partial \tilde{H}}{\partial I_2} \end{array} \right) = \left( \begin{array}{c} \omega_1 (I) \\ \omega_2 (I) \end{array} \right) \), which corresponds to the fact that the frequencies \( \omega_1 \) and \( \omega_2 \) change from term to term.
This is to say, if we consider a Tors with forces \( F_1, F_2 \), and move to a different tors with forces \( F_1', F_2' \), then we have changed the frequencies of the motion.

**Example 3.**

Consider now a motion on a central force, with

Hamiltonian:

\[
H(r, \phi, P_r, P_\phi) = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} + V(r),
\]

where \( (r, \phi) \) are the polar coordinates. Since

\[
\frac{\partial H}{\partial \phi} = 0, \quad \text{then} \quad P_\phi = \text{const}.
\]

Then, we have the two integrals of motion:

\[
T_1 = P_r
\]

\[
T_2 = H(r, P_r, P_\phi) = E.
\]

and the system is integrable.

The two action variables are computed under rotations of \( \phi \) from 0 to \( 2\pi \). Then:

\[
T_1 = \frac{1}{2\pi} \oint_{C_1} P_\phi \, d\phi = \frac{1}{2\pi} \int_{r_2}^{r_1} P_\phi \, dr
\]

\[
T_2 = \frac{1}{2\pi} \oint_{C_2} P_r \, d\phi = \frac{1}{2\pi} \int_{r_1}^{r_2} \sqrt{2m\left(E - V(r) - \frac{\mu^2}{2r^2}\right)} \, dr
\]

\[
= 8\sqrt{5}
\]
The precise form of $I_2$ depends on $V(x)$.

Typical:

\[ H(I_1, I_2) = E \]

\[ I_1 \]

\[ I_2 \]

---

\[ 25. \text{ d Motion on the tori.} \]

So far, we have not proved yet the motion of the types of Hamiltonian systems considered (in general) here, are periodic, we invoke again this assumption and we consider the motion is bounded, and periodic.

Then, in principle, we should be able to express any physical quantity can be expressed as a Fourier series in the angle variables $\psi_k(t)$.

Say we can write:

\[ p(t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} e^{i(k_1\phi_1 + k_2\phi_2 + \cdots + k_u\phi_u)} \]

\[ E_{\phi} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} \left( k_1^2 + k_2^2 + \cdots + k_u^2 \right) \]
where the $Q_k$,...,$Q_n$ are functions of the action variables $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ and can be computed as follows:

$$A^{(n)}(\mathbf{I}) = \int \int \cdots \int q_0(\mathbf{I}, \mathbf{\theta}) e^{i(E_1 I_1 + E_2 I_2 + \cdots + k_n I_n)} \, d\mathbf{I} d\mathbf{\theta}$$

where

$$q_0(t) = q_0(\mathbf{I}, \mathbf{\theta})$$

are the generalized coordinates expressed as functions of $\mathbf{I}$ and $\mathbf{\theta}$.

$\mathbf{I} = (I_1, I_2, \ldots, I_n)$ is the vector of action variables

$\mathbf{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ is the angle variables.

and

$$q_0(t) = \omega_j t + q_j(0)$$

Functions that can be expressed in multiple periodic series are usually called multiply periodic functions or multiple-phase functions. The frequencies are determined by the derivatives $\frac{\partial A(\mathbf{I})}{\partial I_j} = \omega_j$.

If the frequencies are not rationally related, the motion on a particular torus will not repeat. This type of orbits are called quasi-periodic orbits. This means that this particular orbit will cover the torus under consideration.
This can be easily seen on a 2D torus:

Since $\frac{w_2}{w_1} \neq 1$, the trajectory will cover the elemental square $[0, 2\pi] \times [0, 2\pi]$.

* If the frequencies are rationally related, $\frac{w_1}{w_2} \in \mathbb{Q}$, then the motion will be repeated on $[0, 2\pi] \times [0, 2\pi]$.

If, for example, $\frac{w_1}{w_2} = \frac{n_1}{n_2}$, $n_1, n_2 \in \mathbb{N}$, then $n_1 \cdot w_1 = n_2 \cdot w_2$, i.e., $n$ cycles in the $\phi_1$ variable and $m$ cycles in the $\phi_2$ variable, then the orbit will close. Try if we are in the 2D case.

$2 \phi_2 = 1 \cdot 2 \phi_1$

$\Rightarrow n \frac{w_2}{w_1} = 2 \Rightarrow \frac{w_2}{w_1} = 2$  \( \frac{1}{2} \)

and the orbit will be closed.
For the n-dimensional case, to get closed orbits we require:

\[ \sum_{j=1}^{n} k_j w_j = 0 \]

and \( n-1 \) of these equations are required.

**Def:** If a completely integrable system satisfies:

\[
\det \left( \frac{\partial^2 H(T)}{\partial p_i \partial q_j} \right) \neq 0,
\]

it is called **non-degenerate**.

Non-degenerate integrable systems satisfy the properties:

(a) On the energy manifold, some tori are periodic and some other are quasi-periodic (requires a proof).

(b) Since the solutions are not dense in \( \mathbb{R}^n \) and the

\[ \text{nor} \text{-degenerate integrable systems,} \]

quasi-periodic orbits are most likely to happen that

closed orbits. (Requires a proof).

However, closed orbits play a central role in
perturbation theory, in general, and in KAM theory particularly.
2.5.e. General Considerations and Fundamental Questions

In this chapter, we will consider non-integrable systems.

(a) What happens if the system does not have the full set of n integrals of motion?

(b) What happens for a non-integrable Hamiltonian with its Hamilton-Jacobi equations? Do the solutions exist? If so, what do they represent, since they are not a canonical transformation from (q,p) to (Q,I)?

(c) What happens to an integrable Hamiltonian under non-integrable Hamiltonian perturbations? Are the tori preserved, even they are distorted? I.e., even they are diffeomorphic mapped to the perturbed equations?

(d) Given a Hamiltonian system, how can we decide if they are integrable or non-integrable?

These questions are not an easy task to do. Questions (a) and (d) are related to KAM theory (and partially resolved by that theory). Question (b): I do not know.
Question A): Tabor partially solves it in Chapter 8, but this question is still not fully resolved.

Appendix 2 Geometric Concepts in Classical Mechanics

Contravariant Vectors

Consider a vector \( \mathbf{a} = \left( a_1, a_2, ..., a_n \right) \) which is a function of \( x \) of \( \mathbf{x} = (x_1, x_2, ..., x_n) \). Consider the change of variables

\[ \mathbf{x} \rightarrow \mathbf{y} = (y_1, y_2, ..., y_n) \]

Then, we have new functions \( \tilde{a} \) of \( \mathbf{y} \):

\[ \tilde{a} = \tilde{a}(y_1, ..., y_n) = \tilde{a}(x_1(y), x_2(y), ..., x_n(y)) \]

This vector is called contravariant if it transforms under the rule:

\[ \tilde{a}_j = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} a_i \]

(Contravariant vectors are usually written with superscripts:

\[ \tilde{a} = (a^1, a^2, ..., a^n) \]

but Tabor uses the subscripts in his text.)
Example 1. Infinitesimal Translation

\[ dx = (dx_1, \ldots, dx_n) \]

\[ dy_j = \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} \, dx_i \]

Then, the translation is a contravariant vector.

Example 2. Consider a function \( y = y(x) \)

with \( x = x_1, x_2, \ldots, x_n \) and consider the operator:

\[ \xi = \sum_{i=1}^{n} \xi_i \, \frac{\partial}{\partial x_i} \]

where \( \xi_i = \xi_i \, (x_1, x_2, \ldots, x_n) \).

Then

\[ \xi \phi = \sum_{i=1}^{n} \xi_i \, \frac{\partial}{\partial x_i} \phi = \xi \cdot \nabla \phi, \]

i.e.,

\[ \xi \phi = \xi \cdot \nabla \phi. \]

when \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \).

Question: How do the components \( \xi_i \) of \( \xi \)
transform under the change of variables:

\[ (x_1, \ldots, x_n) \rightarrow (y_1, \ldots, y_n), \]

We would like to know how the operator \( \xi \) looks like
in the "y" variables.
This is to say we would like to know how
\[ \vec{\mathbf{e}} = \vec{\mathbf{e}}_0 \nabla \vec{\mathbf{r}} \]
with \[ \vec{\mathbf{e}}_0 = (\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \ldots, \vec{\mathbf{e}}_n). \]

We now have
\[ \vec{\mathbf{e}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{e}}_0 \cdot \nabla \vec{\mathbf{r}} = \sum_{i=1}^{n} \vec{\mathbf{e}}_i \cdot \frac{\partial \vec{\mathbf{r}}}{\partial \vec{x}_i} \]

Since \[ \vec{\mathbf{r}} = \vec{\mathbf{r}}(y_1(x), \ldots, y_n(x)) \]
then
\[ \frac{\partial \vec{\mathbf{r}}}{\partial \vec{x}_i} = \sum_{j=1}^{n} \frac{\partial y_j(x)}{\partial \vec{x}_i} \frac{\partial \vec{\mathbf{r}}}{\partial y_j(x)} \]
and so
\[ = \sum_{i=1}^{n} \vec{\mathbf{e}}_i \sum_{j=1}^{n} \frac{\partial y_j(x)}{\partial \vec{x}_i} \frac{\partial \vec{\mathbf{r}}}{\partial y_j(x)} \]

Thus,
\[ \vec{\mathbf{e}} \cdot \vec{\mathbf{r}} = \sum_{i=1}^{n} \vec{\mathbf{e}}_i \cdot \frac{\partial \vec{\mathbf{r}}}{\partial \vec{x}_i} \]

So \[ \vec{\mathbf{e}}(x) = (\vec{\mathbf{e}}_1(x), \vec{\mathbf{e}}_2(x), \ldots, \vec{\mathbf{e}}_n(x)) \] is a covariant vector.
Example 1.65  Consider the trajectory in the $\mathbb{R}^n$ space

$\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$

The velocity vector is

$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} = \left( \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt} \right)$.

If we change the reference frame from $\mathbf{x}$ to a new reference frame $\mathbf{y}$, of

$\mathbf{y}_0(\mathbf{x}) = \mathbf{y}_0(x_1, x_2, \ldots, x_n)$.

How does look the velocity in this new reference frame then?

We know:

$\frac{d\mathbf{y}_0}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{y}_0}{\partial x_j} \frac{dx_j}{dt} = \mathbf{v} \cdot \mathbf{y}_0$.

or

$\left( \frac{d\mathbf{y}_0}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{y}_0}{\partial x_j} \frac{dx_j}{dt} \right)$

Then

$\mathbf{V} = 
\begin{bmatrix}
\frac{d\mathbf{y}_0}{dt} \\
\frac{d\mathbf{y}_1}{dt} \\
\vdots \\
\frac{d\mathbf{y}_n}{dt}
\end{bmatrix}$

$V_{i} = \sum_{j=1}^{n} \frac{\partial \mathbf{y}_i}{\partial x_j} \frac{dx_j}{dt}$

Then, the components of the velocity $\mathbf{v}$ change as a constant vector. Then, $\mathbf{v}$ is a constant vector.
**Covariant Vectors**

The vector $\tilde{b} = (b_1, \ldots, b_n)$, with $b_j = b_j(x)$, is called a **covariant vector**, if its components change according to the rule:

$$
\tilde{b}_j(\tilde{g}) = \sum_{i=1}^n b_i \frac{\partial x^i}{\partial \tilde{y}_j}.
$$

*(\star)*

**Example:** The gradient field,

$$
b_{j|x} = \frac{\partial F}{\partial x_j} \quad \text{and} \quad \tilde{b}_j(\tilde{g}) = \frac{\partial \tilde{F}}{\partial \tilde{y}_j},
$$

are related as follows: Since

$$
\tilde{F}(\tilde{g}) = F(x, x_1(\tilde{g}), \ldots, x_n(\tilde{g})).
$$

Then:

$$
\frac{\partial \tilde{F}}{\partial \tilde{y}_j} = \sum_{i=1}^n \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial \tilde{y}_j} = \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial x_i} \frac{\partial x_i}{\partial \tilde{y}_j}.
$$

i.e. changes as in $(\star)$ above.

The generalized momentum is the gradient of the action:

$$
P_i = \frac{\partial S}{\partial \dot{x}_i}.
$$

The generalized momentum is a covariant vector.
Then, the generalized vector is a co-variant vector:

\[ \mathbf{\xi} = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \]

is what is called a tangent vector.

If \( \psi = \psi(x_1, x_2, \ldots, x_n) \),

\[ \mathbf{\xi} \psi \] just represents the directional derivative of \( \psi \) at some point \( x = x_0 \):

\[ \lim_{h \to 0} \frac{\psi(x_0 + h) - \psi(x_0)}{h} \]

On the other hand, consider \( \mathbf{\psi}: \mathbb{R} \to \mathbb{R}^n \),

be a curve in \( \mathbb{R}^n \) with parameter \( s \),

\[ \mathbf{\psi}(s) = (\psi_1(s), \psi_2(s), \ldots, \psi_n(s)) \]

with coordinates \( x_j = \psi_j(s) \).

Then, if \( F: \mathbb{R}^n \to \mathbb{R} \), \( F(x_1, \ldots, x_n) \) has derivative along \( \mathbf{\psi}(s) \):

\[ \frac{dF}{ds} = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \frac{dx_i}{ds} \]

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Example 4

As a particular example, the operator is just the derivative
$$\frac{d}{dx}.$$
In particular, for \( F : \mathbb{R}^n \rightarrow \mathbb{R} \), we could have:

\[
F_j = g_j(x_1, \ldots, x_n),
\]

and eqn (97.5) becomes:

\[
\frac{d y_i}{d s} = \sum_{j=1}^{m} \frac{d x_j}{d s} \frac{\partial y_i}{\partial x_j}
\]

\[
= \sum_{j=1}^{m} \frac{\partial y_i}{\partial x_j} \frac{d x_j}{d s}.
\]

So velocities are covariant vectors, and this is equivalent to what was obtained in Example 1, page 97.

**Def.** Consider a point \( x_0 \) on the manifold \( M \) (say \( M \subset \mathbb{R}^m \), and \( dim M = n \) is the manifold on which an integrable system lies); at that point, we have many tangent vectors \( \xi \)'s.

The collection of all these tangent vectors \( \xi \) is called the tangent space of \( M \) at \( x = x_0 \), \( TM_{x_0} \).

**Def.** The collection of all the tangent spaces of \( M \) for all \( x_0 \in M \) is called the tangent bundle.
The Lagrangian description of Mechanics.

We know that, for a Lagrangian description of Mechanics, we require position $q(t)$, and velocity $\dot{q}(t)$.

Then, the tangent vectors:

$$\mathbf{e} = \sum_{i=1}^{n} q_i(\theta), \frac{\partial}{\partial q_i}$$

which represents a point in the tangent bundle.

At time $t$, the particle is at $q(t) \in \mathbb{Q}$, with velocity $\dot{q}(t)$. Thus, the Lagrangian is a mapping from $TM$ to $T\mathbb{R}$:

$$L: TM \rightarrow T\mathbb{R}$$

The Hamiltonian description of Mechanics.

We require:

(1) The generalized positions

$$\mathbf{q} = \{ q_i(t) \}_{i=1}^{n}$$

(2) The generalized (conjugate) momenta

$$\mathbf{p} = \{ p_i(t) \}_{i=1}^{n}$$

where $p(t) = \nabla S$, and they transform as $\nabla$ covariant vectors.
The phase-space is a 2n-dimensional symplectic manifold, and has the property of preserving volumes under Hamiltonian flow. To understand this, the language of differential forms is required.

**Differential One-Forms:**

A differential one-form

\[ w^1 = b_1 \, dx_1 + b_2 \, dx_2 \]

where \( b_1 = b_1(x_1, x_2) \) and \( b_2 = b_2(x_1, x_2) \)

Since \( dx_j \) are the components of contravariant vectors, they by transforms as covariant vectors:

\[ w^1 = \sum_{j=1}^{2} b_j \, dx_j = \sum_{i=1}^{2} \partial x_i \, dy_i = \sum_{i=1}^{2} \left( \sum_{j=1}^{2} b_j \frac{\partial x_j}{\partial y_i} \right) dy_i = \sum_{i=1}^{2} b_i \, dy_i \]

where

\[ b_i = \sum_{j=1}^{2} b_j \frac{\partial x_j}{\partial y_i} \]

Hence \( b_j \) are the components of covariant vectors.

We also observe from above equality that \( w^1 \) is preserved under change of coordinates.

\[ \sum_{j=1}^{2} b_j \, dx_j = w^1 = \sum_{i=1}^{2} b_i \, dy_i = 000 \]
In Hamiltonian mechanics, we require the one-form:

\[ w^j = \sum_{j=1}^{n} p_j \, dq_j \]

to define the action variables. We can also include time \( t \), as another generalized coordinate, with associated conjugate variable, \( \Pi \),

\[ w^j = -\Pi \, dt + \sum_{j=1}^{n} p_j \, dq_j \]

This is called the Poincaré-Cartan 1-form.

Under canonical transformations:

\[ (q, p) \rightarrow (Q, \Pi) \]

This one-form should be invariant (by Liouville's theorem):

\[ -\Pi \, dt + \sum_{j=1}^{n} p_j \, dq_j = -\Pi(i) \, dt + \sum_{j=1}^{n} p_{j(i)} \, dq_{j(i)} \]

where \( H(q, p, t) \), \( \Pi(Q, \Pi, t) \) and \( F \) is the corresponding generating function.

If \( w^j \) is a \( C^1 \) function inside a contour \( \Gamma \), and \( \Gamma \) is contained inside the interior of \( \Gamma' \), then

\[ \oint_{\Gamma} w^j = \oint_{\Gamma'} w^j \]

\[ = \oint_{\Gamma'} w^j \]

\[ = 0 \]
Say $t$ can evolve in time, (if we consider the "extended" phase space). From some $t$ to some $t = T$:

$$\oint \left( \sum_{i=1}^{n} p_i dq_i \right) - H dt = \oint \sum_{i=1}^{n} p_i dq_i - tH dt. \quad (X)$$

Since $dt = 0$, for $t = \text{const}$ and $T = \text{const}$. Therefore:

$$\oint \sum_{i=1}^{n} p_i dq_i = \oint \sum_{i=1}^{n} p_i dq_i \quad (X)$$

The one-form $\sum_{i=1}^{n} p_i dq_i$ is the Poincaré's relative integral invariant.

**Two-forms**

**Differential 2-forms**

The exterior product or wedge product of two differentials are:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

$$dx_i \wedge dx_i = 0,$$

i.e., the obey antisymmetry rules.
They also obey the following rules:
\[ b_i \wedge dx_j = dx_j \wedge b_i = b_p dx_i \]
and so:
\[ dx_i \wedge (b_i dx_j) = b_p dx_i \wedge dx_j \]

Consider now the 1-forms:
\[ \Theta_1 = b_1 dx_1 + b_2 dx_2 \]
\[ \Theta_2 = b_1 dx_1 + c_2 dx_2 \]

Hence:
\[ \Theta_1 \wedge \Theta_2 = (b_1 c_2 - b_2 c_1) dx_1 \wedge dx_2 \]
\[ = - \Theta_2 \wedge \Theta_1 \]

The form \( \Theta_1 \wedge \Theta_2 \) is a 2-form and is a

differential 2-form:
\[ \omega^2 = \Theta_1 \wedge \Theta_2. \]

Under the change of variables,
\[ x_1 = x_1(y_1, y_2) \]
\[ x_2 = x_2(y_1, y_2) \]

we have:
\[ dx_1 = \frac{\partial x_1}{\partial y_1} dy_1 + \frac{\partial x_1}{\partial y_2} dy_2 \]
\[ dx_2 = \frac{\partial x_2}{\partial y_1} dy_1 + \frac{\partial x_2}{\partial y_2} dy_2 \]
\[ dx_1 \wedge dx_2 = \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \left( dy_1 \wedge dy_2 \right) \]
\[ \frac{\partial x_1}{\partial y_1} \frac{\partial x_2}{\partial y_2} \frac{\partial x_1}{\partial y_2} \frac{\partial x_2}{\partial y_1} \]
\[ = \delta(x_1, y_1) dy_1 \wedge dy_2 \]
\[ = \delta(x_1, y_2) dy_1 \wedge dy_2 \]
If the transformation is canonical:

\[ (q_1, q_2) \rightarrow (Q_1, P_1) \]

\[ dq_1 \wedge dp_1 = dQ_1 \wedge dP_1. \]

This can be generalized to any number of variables:

\[ \sum_{i=1}^{n} dq_i \wedge dp_i = \sum_{i=1}^{n} dQ_i \wedge dP_i. \]

And more generally, wedge can be constructed by products:

\[ \omega^2 = \omega^2 \wedge \omega^2 = \sum_{i=1}^{n} dq_i \wedge dp_i. \]

Or, more generally:

\[ \omega^{2n} = \prod_{i=1}^{n} dq_i \wedge dp_i. \]

which is the element of integration in the \( 2n \)-dimensional phase-space.

Two forms can also be obtained from 1-forms by differentiation:

Let \[ w = \sum_{i=1}^{n} b_i dx_i \]

then:

\[ d\omega = \sum_{i=1}^{n} db_i \wedge dx_i. \]

which is a differential 2-form.
Example: Let us consider the 2-dimensional 1-form.

\[ w = b_1(x, y) \, dx + b_2(x, y) \, dy \]

So:

\[
dw = \frac{\partial b_1}{\partial y} \, dx \, dy + \frac{\partial b_2}{\partial x} \, dx \, dy
\]

\[
= \left( \frac{\partial b_1}{\partial x} + \frac{\partial b_1}{\partial y} \right) \, dx \, dy + \left( \frac{\partial b_2}{\partial x} + \frac{\partial b_2}{\partial y} \right) \, dy \, dx
\]

\[
= \frac{\partial b_1}{\partial y} \, dy \, dx + \frac{\partial b_2}{\partial x} \, dx \, dy
\]

\[
= \left( \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} \right) \, dx \, dy
\]

We can now integrate.

\[
\iint_D dw = \iint_D \left( \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} \right) \, dx \, dy
\]

\[
\int_D b_1 \, dx + b_2 \, dy = \iint_D \left( \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} \right) \, dx \, dy
\]

which is Gauss' Theorem, or Stokes' Theorem in higher dimensions. In compact form:

\[
\iint_D w = \iint_D w^2
\]

\[ \text{S} = 105 \]
In Hamiltonian systems,

$$\oint_{\partial A} p_dq = \iint_{A} dp \wedge dq$$

or for a system of \( n \) particles:

$$d \left( \sum_{i=1}^{n} p_i dq_i \right) = \sum_{i=1}^{n} dp_i \wedge dq_i$$

$$\oint_{\partial A} \sum_{i=1}^{n} p_i dq_i = \sum_{i=1}^{n} \iint_{A_i} dp_i \wedge dq_i$$

where \( A_i \) is the area contained in the interior of \( \partial A \) once it is projected into the \((p_1 p_2)\)-plane.

Using (102.\( \ast \)):

$$\sum_{i=1}^{n} \iint_{A_i} dp_i \wedge dq_i = \sum_{i=1}^{n} \iint_{A_i \cap \mathbb{T}} dp_i \wedge dq_i$$

This could be generalized to higher order forms:

$$\iint_{V} \cdots \sum_{i=1}^{n} \frac{1}{V} dp_i \wedge dq_i = \iint_{V} \cdots \sum_{i=1}^{n} \frac{1}{V} dp_i \wedge dq_i$$

and

$$\iint_{V_{\partial A}} \cdots \sum_{i=1}^{n} \frac{1}{V_{\partial A}} dp_i \wedge dq_i = \iint_{V_{\partial A}} \cdots \sum_{i=1}^{n} \frac{1}{V_{\partial A}} dp_i \wedge dq_i$$